Algebra II
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Algebra II
Unit 1: Functions

Time Frame: Approximately five weeks

Unit Description

This unit focuses on development of concepts of functions that were begun in Algebra I and are essential to mathematical growth. This unit explores absolute value expressions and graphs absolute value functions, step functions, and piecewise functions while reviewing linear functions and developing the concepts of composite functions and inverse functions.

Student Understandings

A major goal in the mathematics today is for students to understand functions, be comfortable using numerical, symbolic, graphical, and verbal representations, and be able to choose the best representation to solve problems. In this unit, students review finding the equation of a line in the various forms while developing the concepts of piecewise linear functions, absolute value equations, inequalities, and other functions. Students state their solutions in five forms – reviewing number lines, roster, and set notation begun in Algebra I and developing interval notation and absolute value notation. They also develop the concepts of composite and inverse functions.

Guiding Questions

1. Can students state the difference between a function and a relation in graphical, symbolic, and numerical representations?
2. Can students extend their explanation of the slope of a line to special linear equations such as absolute value, piecewise linear functions, and greatest integer functions?
3. Can students solve absolute value equations and inequalities and state their solutions in five forms when appropriate – number lines or coordinate graphs, roster, set notation using compound sentences using “and” or “or”, interval notation using $\cup$ and $\cap$, and absolute value notation?
4. Can students determine the graphs, domains, ranges, intercepts, and global characteristics of absolute value functions, step functions, and piecewise linear functions both by hand and using technology and verbalize the real world meanings of these?
5. Can students use translations, reflections, and dilations to graph new absolute value functions and step functions from parent functions?
6. Can students find the composition of two functions and decompose a composition into two functions?
7. Can students define one-to-one correspondence, find the inverse of a relation and determine if it is a function?

Unit 1 Grade-Level Expectations (GLEs)

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<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
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<td><strong>Number and Number Relations</strong></td>
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<td>1.</td>
<td>Read, write, and perform basic operations on complex numbers (N-1-H) (N-5-H)</td>
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<td><strong>Algebra</strong></td>
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<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
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<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
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<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
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<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
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<tr>
<td><strong>Geometry</strong></td>
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<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
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<td><strong>Patterns, Relations, and Functions</strong></td>
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<td>24.</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
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<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
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<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
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<td>29.</td>
<td>Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)</td>
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Sample Activities

**Ongoing: Little Black Book of Algebra II Properties**

Throughout the year, have students maintain a math journal of properties learned in each unit. This self-made reference book emphasizes the important concepts in the unit and reinforces the definitions, formulas, real-world applications, and symbol representations. Have students keep these in a black marble composition book (thus the name “Little Black Book”) and grade them for completeness at the end of each unit. Each property should take up approximately one-half page unless the student wishes to add examples for future reference. Have students personalize the title page and paste the list in front of each new unit.
to form a table of contents. A suggested list of properties with possible items for inclusion is provided at the beginning of each unit.

Functions

The following is a list of properties to be included in the Little Black Book of Algebra II Properties. Add other items as appropriate.

1.1 Function of x – define function, how to identify equations as functions of x, how to identify graphs as functions of x, how to determine if ordered pairs are functions of x, an explanation of the meaning of \( f(x) \) (e.g., If \( f(x) = 3x^2 - 4 \), find \( f(3) \) and explain the process used in terms of a function machine).

1.2 Four Ways to Write Solution Sets – explain/define roster, interval notation using \( \cup \), number line, set notation using “and” or “or”.

1.3 Absolute Value Equations and Inequalities as Solution Sets – write solutions in terms of “distance,” change absolute value notation to other notations and vice versa (e.g., write \( |x| < 4, |x - 5| \leq 6, |x| \geq 9 \) as number lines, as words in terms of distance, as intervals, and in set notation; write \([-8, 8], (-4, 6) \) as absolute values).

1.4 Domain and Range – write the definitions, give two possible restrictions on domains based on denominators and radicands, determine the domain and range from ordered pairs, graphs, equations, and inputs and outputs of the function machine; define abscissa, ordinate, independent variable, and dependent variables.

1.5 Slope of a Line – define slope, describe lines with positive, negative, zero and no slope, state the slopes of perpendicular lines and parallel lines.

1.6 Equations of Lines – write equations of lines in slope-intercept, point-slope, and standard forms and describe the process for finding the slope and y-intercept for each form.

1.7 Distance between Two Points and Midpoint of a Segment – write and explain the formula for each.

1.8 Piecewise Linear Functions – define and explain how to find domain and range for these functions.

1.9 Absolute Value Function – define \( y = |x| \) as a piecewise function and demonstrate an understanding of the relationships between the graphs of \( y = |x| \) and \( y = a|x - h| + k \) (i.e., domains and ranges, the effects of changing a, h, and k). For example, write \( y = 2|x-3| + 5 \) as a piecewise function, explain the steps for changing the absolute value equation to a piecewise function, and determine what part of the function affects the domain restrictions.

1.10 Step Functions and Greatest Integer Function – define each and relate to the piecewise function. Graph the functions and find the domains and ranges.

Examples: Solve for \( x: \begin{cases} \frac{1}{2} \frac{x}{2} = 7 \end{cases} \). If \( f(x) = \frac{2x}{5} - 3 \), find \( f(0.6) \) and \( f(10.2) \).

1.11 Composite Functions – define, find the rules of \( f(g(x)) \) and \( g(f(x)) \) using the example, \( f(x) = 3x + 5 \) and \( g(x) = x^2 \), interpret the meaning of \( f \circ g \), explain composite functions in terms of a function machine, explain how to find the domain of composite functions, and how to graph composite functions in the graphing calculator.
1.12 **Inverse Functions** – define, write proper notation, find compositions, use symmetry to find the inverse of a set of ordered pairs or an equation, determine how to tell if the inverse relation of a set of ordered pairs is a function, explain how to tell if the inverse of an equation is a function and explain how to tell if the inverse of a graph is a function.

**Activity 1: Definition of Functions (GLEs: 24, 25)**

In this activity students reinforce the concepts of function verbally, numerically, symbolically and graphically.

**Bellringer:**

*Teacher Note:* Each in-class activity is started with a Bellringer that either reviews past concepts to check for understanding or sets the stage for an upcoming concept. Have the students work the Bellringers in their notebooks preceding the upcoming lesson as you circulate to give individual attention to students who are weak in that area.

Determine if each of the following is a function of \(x\). If not explain the problem/s.

(1) the set of ordered pairs \(\{(x,y) : (1, 2), (3, 5), (3, 6), (7, 5), (8, 2)\}\)
(2) the set of ordered pairs \(\{(s, d) : (1, 1), (2, 4), (3, 9), (-1, 1), (-2, 4), (-3,9)\}\)
(3) the relationship “\(x\) is a student of \(y\)”
(4) the relationship “\(x\) is the biological daughter of mother \(y\)”
(5) the equation \(2x + 3y = 6\)
(6) the equation \(x + y^2 = 9\)
(7) the equation \(y = x^2 + 4\)
(8) the graph of a circle

*Solution:* (1) no, (2) yes, (3) no, (4) yes, (5) yes, (6) no, (7) yes, (8) no

**Activity:**

- Use the Bellringer to ascertain the students’ prior knowledge of functions and to have the students verbalize a definition of a function of \(x\). Several of the definitions may be:
  - A function is a set of ordered pairs in which no first component is repeated.
  - A function is a relation in which, for each value of the first component of the ordered pairs, there is exactly one value of the second component.
  - A function is a relationship between two quantities such that one quantity is associated with a unique value of the other quantity. The latter quantity, often called \(y\), is said to depend on the former quantity, often denoted \(x\).

- Discuss “unique value of the second component” as the key component to functions. The relationship in problem 2 is not a function because, for example, Mary can be a student of Mrs. Joiner and Mr. Black. The relationship in problem 3 is a function because Mary is the biological daughter of only one woman.

- Discuss how to tell if ordered pairs, equations, and verbal descriptions are functions.
• **Function Machine:** Paint a visual picture using a *function machine*, which converts one number, the input, into another number, the output, by a rule (equation, sentence, set) in such a manner that each input has only one output. Define the input as the independent variable and the output as the dependent variable.

| input 5 | Rule: \( x^2 + 3 \) | output 28 |

• Have students write the rule (equation) that symbolizes the relationship of the following and draw a function machine for an input of 4 in the following situations:

1. The area of a circle depends on its radius
2. The length of the box is twice the width thus the length depends on the width.
3. The state tax on food is 5% and the amount of tax you pay depends on the cost of the food bought.
4. \( d \) depends on \( s \) in Bellringer #2

   *(Solution:)*

   | (1) \( r = 4 \) | Rule: \( \pi r^2 \) | \( A = 16\pi \text{ cm}^2 \) |
   | (2) \( w = 4 \) | Rule: \( 2w \) | \( l = 8 \text{ in.} \) |
   | (3) \( c = 4 \) | Rule: \( .05c \) | \( t = $.20 \) |
   | (4) \( s = 4 \) | Rule: \( s^2 \) | \( d = 16 \) |

• **Functions symbolically:** Discuss function notation. When the function \( f \) is defined with a rule or equation using \( x \) for the independent variable and \( y \) for the dependent variable, the terminology “\( y \) is a function of \( x \)” is used to emphasize that \( y \) depends on \( x \) and is denoted by the notation \( y = f(x) \). (Make sure to remind the students that the parentheses do not indicate multiplication.) Stress that the symbolism \( f(3) \) is an easy way to say “find the \( y \)-value that corresponds to an \( x \)-value of 3”.

   o Using the function machines above, have students rewrite the rules in function notation defining the functions \( A(r), l(w), t(c), \text{and } d(s) \)

   *(Solutions:)* (1) \( A(r) = \pi r^2 \), (2) \( l(w) = 2w \), (3) \( t(c) = .05c \), (4) \( d(s) = s^2 \)

   o Using Bellringers 5 and 7, rewrite \( y \) as \( f(x) \).

   *(Solutions:)* (5) \( f(x) = -\frac{2}{3}x + 2 \), (7) \( f(x) = x^2 + 4 \)

   o Using Bellringer 6, have students determine why they cannot write \( y \) as a function of \( x \).

   *(Solution:)* When you isolate the \( y \) there are two outputs: \( y = +\sqrt{9-x} \) and \( y = -\sqrt{9-x} \).

   | \( x = 4 \) | Rule: \( \pm\sqrt{9-x} \) |

• **Functions Graphically:** In Bellringer 8, there is no rule or set of ordered pairs, just a graph. Have the students develop the vertical line test for functions of \( x \).
Lead a discussion of the meaning of \( y = f(x) \) which permits substituting \( x \) for all independent variables and \( y \) for all dependent variables. Have the students use a graphing calculator to graph the functions developed above: \( A(r) \) graphed as \( y = \pi x^2 \), \( l(w) \) graphed as \( y = 2x \), \( t(c) \) graphed as \( y = .05x \), and \( d(s) \) graphed as \( y = x^2 \). Have students determine if the relations pass the vertical line test.

- **Critical Thinking Writing Activity:**

  *Teacher Note: This is the first of many Critical Thinking Writing activities. Post the grading rubric that will be used throughout the year on the wall. This rubric can be found in the Activity-Specific Assessments of this unit.*

  The relationship between the profit, \( f(x) \), in dollars made on the sale of a book sack and the cost, \( x \), in dollars, of that book sack is given by the function \( f(x) = -2x^2 + 4x + 5 \). What is the value of \( f(2) \)? Have students describe in their own words what \( x = 1 \) and \( f(1) \) mean in real world terms, using the terminology independent and dependent variable. Ask students to graph the function on their graphing calculators, sketch the graph on paper and explain what is happening to the relationship between cost and profit.

  *Solution: 1 is the independent variable, cost, and \( f(1) \) is the dependent variable, profit. The profit depends on the cost, so if a book sack costs $1.00 to make, the profit will be $7.00. As the cost of the book sack increases to $1, the most profit of $7.00 is made, but after that less profit is made. (Answers may vary.)*

**Activity 2: Interval and Absolute Value Notation (GLEs: 1, 8, 10, 25)**

This activity reviews how to express answers in roster and set notation and teaches interval and absolute value notation. Linear functions are taught extensively in Algebra I but should be continuously reviewed. In this activity students will review graphing linear functions using interval notation.

**Bellringer:** Have students draw the following on a number line:

1. \( x \in \{ -2, 1, 3, 4 \} \)
2. \( \{ x : x > 4 \} \)
3. \( \{ x : x < 2 \text{ or } x \geq 5 \} \)
4. \( \{ x : x > 3 \text{ and } x < 0 \} \)
5. \( \{ x : x \geq 5 \text{ and } x < 8 \} \)
6. \( \{ x : -3 \leq x < 6 \} \)
7. \( \{ x \in \mathbb{R} \} \)

**Activity:**

- Use the Bellringer to review three of the five ways to write solution sets:
  
  1. **Roster Notation:** Use when the solutions are finite or can be infinite if a pattern exists such as \( \{ \ldots, 2, 4, 6, \ldots \} \); however the answers are discrete and not continuous. The
three dots are called ellipsis and represent numbers that are omitted, but the pattern is understood.

2. **Set Builder Notation**: Use when the answers are continuous and infinite. Review the use of the words **and** for intersection and **or** for union. Discuss that the notation in Bellringer 6 is an **and** situation similar to Bellringer 5. Ask the students to identify the difference in the set notation \( \{ x : 0 > x > 3 \} \) and notation used in Bellringer 4.

3. **Number Line**: Use with roster notation using closed dots or set notation using solid lines. In Algebra I an open dot for endpoints that are not included such as in \( x > 2 \) and a closed dot for endpoints that are included such as in \( x \geq 2 \) were used. Introduce the symbolism in which a parenthesis “( )” represents an open dot and a bracket “[ ]” represents the closed dot on a number line. Use this notation to draw the number line answers for Bellringers 3, 5, and 6.

- **Introduce Interval Notation**: Use intervals to write continuous, infinite sets with the following guidelines:
  1. Bracket – indicates that the endpoint is included. Never use brackets with infinity
  2. Parenthesis – indicates that the endpoint is not included
  3. **∪** and **∩** – Use the symbol **∪**, union, for **or** statements and **∩**, intersection, for **and** statements. Most **and** statements can be written as one interval and rarely use **∩**. For example, since Bellringer 3 has no solution the interval notation would be \( \emptyset \). Since Bellringer 5 is between 5 and 8 the interval \([5, 8)\) is more common and simpler than using \((−∞, 8] \cap [5, ∞)\).

Have the students rewrite all the Bellringers in interval notation.

**Solutions:** (1) Cannot use interval notation - the set is not continuous, (2) \((4, ∞)\), (3) \((-∞, 2) ∪ [5, ∞)\), (4) \(\emptyset\), (5) \([5, 8]\), (6) \([-3, 6)\), (7) \((-∞, ∞)\)

- **Introduce Absolute Value Notation**: Review the absolute value concepts from Algebra I.
  - **Absolute Value Equalities**: Define \( |a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases} \); therefore \( |5| = 5 \) and \( |-5| = 5 \).
  - Have students solve \( |x| = 8 \) and list the answers in set builder notation and roster notation. **Solution:** \( \{ x : x = 8 \text{ or } x = -8 \} \), \( \{8, -8\} \)
  - **Absolute Value as Distance**: Define absolute value as the distance on a number line from a center point. For example, \( |x| = 5 \) can be written verbally as, “This set includes the two numbers that are a distance of 5 from zero.” Have students express the following absolute value equalities in roster notation, set builder notation, on the number line, and verbally as distance.
(1) \( |x| = 7 \)
   Solution: \( \{7, -7\}, \{x : x = 7 \text{ or } x = -7\} \). This set includes the two numbers that are equal to a distance of 7 from zero.

(2) \( |x + 2| = 8 \).
   Solution: \( \{-10, 6\}, \{x : 7 = -10 \text{ or } x = 7\} \). This set includes the two numbers that are equal to a distance of 8 from -2.

(3) \( |x - 4| = 5 \).
   Solution: \( \{-1, 9\}, \{x : x = -1 \text{ or } x = 9\} \). This set includes the two numbers that are equal to a distance of 5 from 4.

After working the examples have students develop the formula \( |x - h| = d \) where \( h \) is the center and \( d \) is the distance.

- **Absolute Value Inequalities:**
  - Develop the meaning of \( |a| < b \) from the definition of absolute value:
    \[ |a| < b \Rightarrow a < b \text{ and } -a < b \therefore a < b \text{ and } a > -b. \]
  - Develop the meaning of \( |a| > b \) from the definition of absolute value:
    \[ |a| > b \Rightarrow a > b \text{ or } -a > b \therefore a > b \text{ or } a < -b. \]
  - Replacing the “\( = \)” in the previous examples with inequalities, have students express the following absolute value inequalities in set builder notation, on the number line, verbally as distance, and in interval notation:
    (1) \( |x| \leq 7 \) Solution: \( \{x : -7 \leq x \leq 7\} \). This set includes all numbers that are less than or equal to a distance of 7 from 0. \([-7, 7]\).
(2) \(|x + 2| > 8\) Solution: \(x: x < -10 \text{ or } x > 6\), This set includes all numbers that or greater than a distance of 8 away from –2. \((-\infty, -10) \cup (6, \infty)\).

(3) \(|x - 4| \leq 5\) Solution: \(x : -1 < x < 9\), This set includes all numbers that are less than a distance of 5 from 4. \([-1, 9]\).

- Use Interval Notation to Review Graphs of Linear Functions

(1) Give students the graph of a line with endpoints \((-1, 2)\) and \((3, 7)\). Ask students to write the equation of the line and the values of \(x\) and \(y\) written in interval notation.

\[ Solution: (1) \ y = \frac{5}{4}x + \frac{13}{4} \quad x: [-1, 3], \ y: [2, 7] \]

(2) Give students the equation \(f(x) = 3x - 6\) and ask them to graph it on the interval, \([-3, 1]\). Discuss whether this is an interval of \(x\) or \(y\).

Activity 3: Domain and Range (GLEs: 1, 4, 6, 8, 10, 24, 25)

The focus of this activity is the use of roster, interval, and absolute value notations to specify the domain and range of functions from ordered pairs, equations, and graphs.

Bellringer: Have students graph the following functions on the graphing calculator, adjust the window to find both intercepts, sketch the graph in their notebook, and find \(f(0)\):  
(1) \(f(x) = 2x + 12\)  
(2) \(f(x) = x^2 + 23\)  
(3) \(f(x) = \sqrt{x - 5}\)  
(4) \(f(x) = \frac{1}{x}\)
Activity:

- Use the Bellringer to review steps for using the features of the graphing calculator such as graphing, changing the window, and finding \( f(0) \) in three ways:
  1. trace by moving the cursor (the most inaccurate because of pixels)
  2. trace by typing in \( x = \) the value
  3. typing \( y_1(0) \) on the home screen

- Domain and Range from real-world applications

  Consider the functions from Activity 1:
  1. The area of a circle depends on its radius, \( A(r) = \pi r^2 \).
  2. The length of the box is twice the width thus depends on the width, \( l(w) = 2w \).
  3. The state tax on food is 5%, and the amount of tax you pay depends on the cost of the food bought, \( t(c) = 0.05c \).
  4. \( d \) depends on \( s \) in a set of ordered pairs, \( \{(s, d) : (1,1), (2,4), (3,9), (-1,1), (-2,4), (-3,9)\} \).

  - Have students determine the independent variables and discuss what numbers can be used in the real world situation for these independent variables.
  - **Solution:** (1) all radii \( \geq 0 \), (2) all widths \( > 0 \) (3) all costs \( \geq 0 \) (4) only the numbers 1, 2, 3, -1, -2, and -3

  - Define **domain** as “the allowable values of the independent variable.”

  - Have students determine the results for the dependent variable when using the restricted independent variable.
  - **Solution:** (1) all Areas \( \geq 0 \), (2) all lengths \( > 0 \), (3) all taxes \( \geq 0 \), (4) only the numbers 1, 4, and 9.

  - Define **range** as “the resulting values of the dependent variable.”

  - Have students list the domain and range of the four examples using interval notation, if possible.
  - **Solution:** (1) \( D: [0, \infty), R [0, \infty) \) (2) \( D: (0, \infty), R: (0, \infty) \) (3) \( D: [0, \infty), R [0, \infty) \) (4) cannot write as interval only roster \( D: \{1, 2, 3, -1, -2, -3\}, R: \{1, 4, 9\} \)

- Domain and range from graphs

  Discuss determining the domain and range from graphs. Give the following graphs to practice writing in interval notation (problems 1-6) and in absolute value notation (problems 2 and 5). Assume the graph continues to infinity as the picture leaves the screen.

  *Teacher Note: The graphics shown on the next page and other similar graphics in this document were generated by a TI-83® calculator.*
Solution:

*Interval Notation:*

1. $D: (-\infty, \infty) \quad R: [4, \infty)$
2. $D: (-\infty, 0) \cup (0, \infty) \quad R: (-\infty, 0) \cup (0, \infty)$
3. $D: [0, \infty) \quad R: (-\infty, 5)$
4. $D: (-\infty, \infty) \quad R: [-8, \infty)$
5. $D: [-3, 3] \quad R: [0, 3]$
6. $D: (-\infty, -1] \cup [2, \infty) \quad R: [-3, \infty)$

*Absolute Value Notation:*

2. $D: |x| > 0 \quad R: |y| > 0$
5. $D: |x| \leq 3, \quad R: |y - 1.5| \leq 1.5$
• Give the students the following problems and ask if there are any values of \(x\) that are not allowed therefore creating a restricted domain. Write the domains in set notation.

\[
\begin{align*}
(1) \quad f(x) &= \frac{1}{x} \\
(2) \quad g(x) &= \sqrt{x} \\
(3) \quad f(x) &= \frac{1}{2x-6} \\
(4) \quad g(x) &= \sqrt{x-2}
\end{align*}
\]

**Solution:**
\[
\begin{align*}
(1) \quad \{x : x \neq 0\} \\
(2) \quad \{x : x \geq 0\} \\
(3) \quad \{x : x \neq 3\} \\
(4) \quad \{x : x \geq 2\}
\end{align*}
\]

• Develop two types of domain restrictions in the real number system:
(1) Division by zero is undefined
(2) The value under the square root must be \(> 0\).
Provide problems in which students find these two types of domain restrictions.

• **Combination of functions**
Given \( f(x) = \sqrt{x-2} \) and \( g(x) = \frac{1}{x-3} \), have students evaluate the following and determine the domain of the final function in set notation:
\[
\begin{align*}
(1) \quad (f + g)(x) \\
(2) \quad (fg)(x) \\
(3) \quad \frac{g}{f}(x)
\end{align*}
\]

**Solution:**
\[
\begin{align*}
(1) \quad (f + g)(x) &= \sqrt{x-2} + \frac{1}{x-3}, \quad \text{Domain:} \{ x \geq 2, \ x \neq 3 \} \\
(2) \quad (fg)(x) &= \frac{\sqrt{x-2}}{x-3}, \quad \text{Domain:} \{ x \geq 2, \ x \neq 3 \} \\
(3) \quad \frac{g}{f}(x) &= \frac{1}{(x-3)\sqrt{x-2}}, \quad \text{Domain:} \{ x > 2, \ x \neq 3 \}
\end{align*}
\]

• **Critical Thinking Writing Activity**
Consider the relationship between the profit, \( f(x) \), in dollars made on the sale of a booksack and the cost, \( x \), in dollars, of that booksack given by the function, \( f(x) = -2x^2 + 4x + 5 \) in the Activity 1 assignment. Have students discuss the real-world domain and range of this function.
Solution: The domain would be the cost. It definitely has to cost more than $0 to make the book sacks, but you don’t want to lose money, so you should cut off the domain when the profit is negative. The domain must be $0, 4$. The range would be the profit and if the cost stays between 0 and 4, then the profit is at least 0 but maxes out at $7.00. The range would be $[0, 7]$.

Activity 4: Solving Absolute Value Equations and Inequalities (GLEs: 1, 8, 24, 29)

The focus for this activity is solving more absolute value equations and inequalities and expressing solutions in interval and set notation.

Bellringer: Have students write the solutions for the following absolute value equations in interval or roster notation and in terms of distance.

(1) $|x – 2| = 3$
(2) $|x + 3| \leq 4$
(3) $|x – 6| > 5$
(4) $|2x + 6| = 10$
(5) $|3x – 9| = –3$

Solutions:
(1) $\{5, –1\}$, The solutions are equal to a distance of 3 from 2.
(2) $[–7, 1]$, All the solutions are less than or equal to a distance of 4 from –3.
(3) $(-\infty, 1) \cup (11, \infty)$ The solutions are more than a distance of 5 from 6.
(4) $\{–8, 2\}$. The solutions are equal to a distance of 5 from –3.
(5) the empty set

Activity:
- Use Bellringer problems 1 – 3 to review notations from Activity 2.
- Absolute Value Equalities
  - Have the students discuss the procedure they used to solve Bellringer problem 4.
  - Review the definition – “what is inside the absolute value signs is positive and negative.” The progression of steps in solving an absolute value is important for future work with absolute values.

Solve: $|2x + 6| = 10$

Solution:
$2x + 6 = 10 \text{ or } -(2x + 6) = 10$ (Do not allow students to skip this step.)

$2x = 4 \Rightarrow x = 2$
$2x = –10 \Rightarrow x = –5$

$2x = –16 \Rightarrow x = –8$

- Stress that students should think of the big picture first when attempting to solve Bellringer problem 5 (i.e., absolute values cannot be negative therefore the answer is the empty set).
• **Properties of Absolute Value Expressions**

Have the students decide if the following equations are true or false, and if false give counter-examples:

Property 1: \[|ab| = |a||b|\]
Property 2: \[|a + b| = |a| + |b|\]

*Solution:* (1) true (2) false |2 + (−5)| < |2| + |−5|

o Ask the students how they would use Property #1 (above) to help solve the problem |2x + 6| = 10?

*Solution:* |2x + 6| = 10 \[\Rightarrow 2|x + 3| = 10 \Rightarrow |x + 3| = 5 \Rightarrow \text{the distance from −3 is 5; therefore, the answer is −8 and 2.}"

• **Absolute Value Inequalities**

o Ask the students to solve and discuss how they can think through the following problems to find the solutions instead of using symbolic manipulations:

1) \[|3x – 15| < 24\]
2) \[|5 – 2x| > 9\]

Develop the notion that “is equal to” is the division between “is greater than” and “is less than”; therefore, changing these inequalities into equalities to find the boundaries on the number line and then choosing the intervals that make the solution true are valid processes to use.

*Solution:*

(1) The boundaries occur at \(x = −3\) and 13 and the interval between these satisfies the inequality so the answer is \(−3 < x < 13\)
(2) The boundaries occur at −2 and 7 and the intervals that satisfy this equation occur outside of these boundaries so the answer is \(x < −2\) or \(x > 7\).

o Relate these answers to the distance concepts in Activity 2 and use the discussion to develop the rules for \(>\) (or relationship) and \(<\) (and relationship) and why.

• **Critical Thinking Writing Activity**

The specifications for machined parts are given with tolerance limits. For example, if a part is to be 6.8 cm thick, with a tolerance of .01 cm, this means that the actual thickness must be at most .01 cm, greater than or less than 6.8 cm. Between what two thicknesses is the dimension of the part acceptable? Show your work and write an absolute value equation that models this situation and discuss why absolute value should be used here.

*Solution:* \(|x – 6.8| ≤ 0.01\) *Explanations will vary.*

**Activity 5: Linear Functions (GLEs: 1, 4, 6, 16, 25, 28, 29)**

There are no GLEs in 11th and 12th grade for graphs of lines, but it is imperative for future mathematical growth that students spend time reviewing properties of equations and increase the speed and accuracy with which they draw graphs of lines. This activity focuses on transforming linear equations into linear functions along with a discussion of function notation and domain and range.
Bellringer: Have students work with a partner to copy the following terms, write the formula, and make up and work a problem using the concept.

1. slope of a line
2. slope of horizontal line
3. equation of a horizontal line
4. slope of a line that starts in Quadrant III and ends in Quadrant I
5. slope of a line that starts in Quadrant II and ends in Quadrant IV
6. slope of vertical line
7. equation of a vertical line
8. slopes of parallel lines
9. slopes of perpendicular lines
10. standard form of equation of line
11. \( y \)-intercept form of equation of line
12. point-slope form of equation of line
13. distance formula
14. midpoint formula

Activity:
- Let each pair of students choose one of the concepts and put their problem (unworked) on the board for everyone else to work. After all students have worked the problems, then have the pair explain it. If the students are having problems with a particular concept, choose another pair’s problem for everyone to work.
- **Point-slope form of the equation of a line** is one of the most important forms of the equation of a line for future mathematics courses such as Calculus. Have pairs of students find the equations of lines for the following situations using point-slope form without simplifying:
  1. slope of 4 and goes through the point \((2, -3)\)
  2. passes through the two points \((4, 6)\) and \((-5, 7)\)
  3. passes through the point \((6, -8)\) and is parallel to the line \(y = 3x + 5\)
  4. passes through the point \((-7, 9)\) and is perpendicular to the line \(y = \frac{1}{2}x + 6\)
  5. passes through the midpoint of the segment whose endpoints are \((4, 8)\) and \((-2, 6)\) and is perpendicular to that segment

  **Solution:**
  
  \[
  \begin{align*}
  (1) \quad y + 3 &= 4(x - 2), \\
  (2) \quad y - 6 &= -\frac{1}{2}(x - 4), \\
  (3) \quad y + 8 &= 3(x - 6), \\
  (4) \quad y - 9 &= -2(x + 7), \\
  (5) \quad y - 7 &= -3(x - 1).
  \end{align*}
  \]

  or \(y - 7 = -\frac{1}{2}(x + 5)\)

- **Graphing lines**
  Have the pairs of students use the above problems and their graphing calculators to:
  o Isolate \(y\) in each of the above equations (without simplifying)

  *Teacher Note: Writing the equation in this form begins the study of transformations that is a major focus in Algebra II for all new functions.*
Solutions:
(1) \( y = 4(x - 2) - 3 \)  
(2) \( y = -\frac{1}{9}(x - 4) + 6 \)  
(3) \( y = 3(x - 6) - 8 \)  
(4) \( y = -2(x + 7) + 9 \)  
(5) \( y = -3(x - 1) + 7 \)

- Graph each equation on the calculator, adjusting the window to see both intercepts. Sketch a graph in your notebook labeling the intercepts.
- Trace all of the points specified in the problem to make sure the equation is entered correctly.
- Simplify each of the equations above and replace \( y \) with \( f(x) \). Find \( f(0) \).

Solution:
(1) \( f(x) = 4x - 11 \), \( f(0) = -7 \)
(2) \( f(x) = -\frac{1}{9}x + \frac{58}{9} \), \( f(0) = \frac{58}{9} \)
(3) \( f(x) = 3x - 26 \), \( f(0) = -26 \)
(4) \( f(x) = -2x - 5 \), \( f(0) = -5 \)
(5) \( f(x) = -3x + 10 \), \( f(0) = 10 \)

- Compare the answers written in \( f(x) \) form to the \( y \)-intercept form, identify the slope in both forms, and discuss the relationship between the \( y \) intercept and \( f(0) \).

• Translating Graphs of Lines – Discovery Worksheet:
Graph each set of three lines on the same screen on your graphing calculator. Discuss the changes in the equation and what affect the change has on the graph.

1. \( f(x) = 2(x - 0) + 0 \)  
   \( g(x) = 2(x - 4) + 0 \)  
   \( h(x) = 2(x + 5) + 0 \)

2. \( f(x) = 2(x - 0) + 0 \)  
   \( g(x) = 2(x - 0) - 4 \)  
   \( h(x) = 2(x - 0) + 5 \)

3. \( f(x) = 2(x - 0) - 4 \)  
   \( g(x) = \frac{1}{2}(x - 0) - 4 \)  
   \( h(x) = 4(x - 0) - 4 \)

4. \( f(x) = 2(x - 0) + 5 \)  
   \( g(x) = -2(x - 4) + 5 \)  
   \( h(x) = 0(x - 5) + 5 \)

5. \( f(x) = 3x + 6 \)  
   \( g(x) = -3x + 7 \)  
   \( h(x) = 3x - 5 \)

• Critical Thinking Writing Activity
Consider the linear function \( f(x) = A(x - B) + C \). Discuss all the changes in the graph as \( A \), \( B \), and \( C \) change from positive, negative, zero and get smaller and larger. Discuss the domain and range of linear functions.
Activity 6: Piecewise Linear Functions (GLEs: 6, 8, 10, 16, 24, 25, 28, 29)

In this activity, the students will review the graphs of linear functions by developing the graphs for piecewise linear functions.

Bellringer: Graph the following:

1. \( g(x) = 2x + 4 \)
2. \( h(x) = -3x - 9 \)

Activity:

- Use the Bellringer to review the y-intercept form of an equation. Make sure students are proficient in graphing lines quickly.

- Give students the definition of a piecewise function—a function made of two or more functions and written as 
  \[ f(x) = \begin{cases} 
  g(x) & \text{if } x \in \text{Domain 1} \\
  h(x) & \text{if } x \in \text{Domain 2} 
  \end{cases} \]

  where Domain 1 \( \cap \) Domain 2 = \( \emptyset \) and Domain 1 \( \cup \) Domain 2 = All real numbers.

- Add the following restrictions to the domains of the functions in the Bellringer problems 1 and 2. Have students regraph \( g(x) \) with a domain of \( x > -2 \) and \( h(x) \) with a domain of \( x \leq -2 \) on the same graph with the starting point labeled.

- Have students rewrite \( f(x) \) as a piecewise function of \( g(x) \) and \( h(x) \).

  **Solution:** 
  \[ f(x) = \begin{cases} 
  2x + 4 & \text{if } x > -2 \\
  -3x - 9 & \text{if } x \leq -2 
  \end{cases} \]

- Discuss whether \( f(x) \) is a function and find the domain and range.

  **Solution:** yes it is a function with \( D: \) all reals, \( R: x \geq -9 \)

- Guided Practice: Have the students graph the following two problems and find the domains, ranges, and zeroes. Calculate \( f(3), f(1), f(0), g(4), g(0), \) and \( g(-2) \) for each.

  1. \( f(x) = \begin{cases} 
  3x + 2 & \text{if } x > 1 \\
  -4x - 2 & \text{if } x \leq 1 
  \end{cases} \)

  2. \( g(x) = \begin{cases} 
  -\frac{3}{4}x - 1 & \text{if } x \geq 0 \\
  -2x & \text{if } x < 0 
  \end{cases} \)

  **Solutions:**
  1. Domain: All reals, range: \( y \geq -6 \), zeroes: \( x = -\frac{1}{2}, f(3) = 11, f(1) = -6, f(0) = -2 \)
  2. Domain: All reals, range: \( (-\infty, -1] \cup (0, \infty) \), zeroes: none, \( g(4) = -4, g(0) = -1, g(-2) = 4 \)

- Critical Thinking Writing Activity

  The market for domestic cars in the US reported the following data. In 1993, 73% of US cars were domestic and in 1996, 72% and in 1999, 69% were domestic. Using these three data points \((3, 73), (6, 72), \) and \((9, 69)\), find two linear equations for the line.
segments and write as a piecewise function. Discuss which interval was the greatest decline and why? Find \( f(7) \) and discuss what \( f(7) \) means in real world terms.

Solution: The decline was the greatest from 1996 to 1999.

\[
f(x) = \begin{cases} 
\frac{1}{3}x + 74 & \text{if } 3 \leq x \leq 6 \\
-\frac{1}{3}x + 78 & \text{if } 6 \leq x \leq 9
\end{cases}
\]

\( f(7) = 71 \) meaning in 1997, 71% of cars were domestic.

Activity 7: Absolute Value Functions (GLEs: 1, 4, 6, 8, 10, 16, 24, 25, 28, 29)

In this activity, students will develop the graphs of absolute value functions from piecewise functions and continue their development of translating functions based on constants. They will change absolute value functions into piecewise functions and relate graphing absolute value functions to solving absolute value inequalities.

Bellringer: Graph the following piecewise functions:

1. \( f(x) = \begin{cases} 
x & \text{if } x \geq 0 \\
-x & \text{if } x < 0
\end{cases} \)

2. \( g(x) = \begin{cases} 
x - 1 & \text{if } x \geq 1 \\
-x + 1 & \text{if } x < 1
\end{cases} \)

Activity:

- Discuss whether Bellringer problem 1 is a function and find the domain and range of \( f(x) \).
  
  Solution: Yes, it is a function with \( D: \text{all reals} \) and \( R: x \geq 0 \)

- Have students graph \( y = |x| \) on their graphing calculators and discuss its relationship to Bellringer problem 1 and the definition of absolute value from Activity 2. Discuss the shape of the graph, slope of the two lines that create the graph, the vertex, the domain and range, and the axis of symmetry.

- Have students graph \( y = |x - 1| \) on their graphing calculators and discuss its relationship to the graph in Bellringer problem 2 and the definition of absolute value from Activity 2. Discuss the shape of the graph, slope of the two lines that create the graph, the vertex, the domain and range, and the axis of symmetry.

- Translating Graphs Absolute Value Functions in the form \( f(x) = a|x - h| + k \)

  Arrange the students in groups and give them the following problems. Have students graph the following equations on their graphing calculators:

  1. \( y = |x - 2| \),
  2. \( y = |x + 3| \),
  3. \( y = |x| + 4 \),
  4. \( y = |x| - 6 \),
  5. \( y = 2|x| \),
  6. \( y = \frac{1}{2} |x| \)
  7. \( y = -3|x| \)
• **1st Critical Thinking Writing Activity**
Consider the Absolute Value Function, \( f(x) = a|x - h| + k \). Discuss all the changes in the graph of the absolute value function as \( a, h, \) and \( k \) change from positive, negative, or zero and get smaller and larger. Discuss the vertex, the equation of the axis of symmetry, whether it opens up or down, how to find the slope of the two lines that make the “V,” and the domain and range of absolute value functions.

• Have students graph the function \( f(x) = 3|x - 2| + 4 \) without their calculators and then check with their calculators, adjusting the window to find all intercepts. Ask them to locate the vertex and equation of the axis of symmetry, state the domain and range, determine the slopes of the two lines that form the “V” and find the \( x-\) and \( y-\) intercept. 

**Solution:** (9) vertex: \((2, 4)\), axis of sym. \( x = 2 \), domain: all reals, range: \( y \geq 4 \), slopes are \( \pm 3 \), no \( x-\)intercept, \( y-\)intercept \((0, 10)\)

• **Changing Absolute Value Functions to Piecewise Functions:**
  
  - Have the students draw the graph of \( f(x) = 3|x - 2| + 4 \) accurately on graph paper and extend both lines to intercept the \( x \) and \( y-\)axis. Have them find the two functions \( g(x) \) and \( h(x) \) that are the equations for the lines that create the “V” and check by graphing the two lines on the calculator to see if they coincide with the \( f(x) \) graph. Have them determine the domain restrictions to cut off the lines at the vertex, then write \( f(x) \) as a piecewise function of the two lines:  

  \[
  f(x) = \begin{cases} 
  3x - 2 & \text{if } x \geq 2 \\
  -3x + 10 & \text{if } x < 2 
  \end{cases}
  \]

  - Have students develop the following steps to symbolically create a piecewise function for an absolute value function without graphing: 

  1. Remove the absolute value signs and replace with the parenthesis keeping everything else for \( g(x) \) and simplify. 
  2. Do the same for \( h(x) \), but put a negative sign in front of the parenthesis and simplify. 
  3. Determine the domain by the horizontal shift inside the absolute value, – shifts right and + shifts left, and put the equal on ≥.)

  - Guided Practice: Have students change to a piecewise functions and check on graphing calculator:

  (1) \( f(x) = |x| + 4 \)  
  (2) \( f(x) = 2|x + 4| \)  
  (3) \( f(x) = -4|x| + 5 \)  
  (4) \( f(x) = -2|x - 4| + 5 \)

  **Solutions:**

  (1) \( f(x) = \begin{cases} 
  x + 4 & \text{if } x \geq 0 \\
  -x + 4 & \text{if } x < 0 
  \end{cases} \)  
  (3) \( f(x) = \begin{cases} 
  -4x + 5 & \text{if } x \geq 0 \\
  4x + 5 & \text{if } x < 0 
  \end{cases} \)  
  (2) \( f(x) = \begin{cases} 
  2x + 8 & \text{if } x \geq -4 \\
  -2x - 8 & \text{if } x < -4 
  \end{cases} \)  
  (4) \( f(x) = \begin{cases} 
  -2x + 13 & \text{if } x \geq 4 \\
  2x - 3 & \text{if } x < 4 
  \end{cases} \)
• **2nd Critical Thinking Writing Activity**
  1. Solve the inequality $|x - 4| \geq 5$ and write as a distance.
  2. Isolate zero in this example.
  3. Graph the function $f(x) = |x - 4| - 5$.
  4. Write the function as a piecewise function and find the zeroes of each piece.
  5. Use the graph of $f(x)$ to determine the interval notation where $f(x) \geq 0$.
  6. Use the graph of $f(x)$ and the piecewise function for $f(x)$ to explain why the solution to $|x - 4| \geq 5$ is an “or” statement instead of an “and” statement.

  *Solution:*
  
  (1) $x$ is a distance less than or equal to 5 units from 4 so the answer is $-1 \leq x \leq 9$.
  
  (2) $|x - 4| - 5 \leq 0$
  
  (4) $f(x) = \begin{cases} x - 9 & \text{if } x \geq 4 \\ -x + 1 & \text{if } x < 4 \end{cases}$ with zeroes at $x = 1$ and 9
  
  (5) $(-\infty, 1) \cup [9, \infty)$

**Activity 8: Step Functions (GLEs: 1, 4, 6, 8, 10, 16, 24, 25, 28, 29)**

In this activity, the students will discover the applications of step functions and how to graph step functions as well as how to graph and write the piecewise function for greatest integer functions.

Bellringer: Graph $f(x) = \begin{cases} 5 & \text{if } x \geq 3 \\ 2 & \text{if } -4 \leq x < 3 \\ -3 & \text{if } x < -4 \end{cases}$. Find domain, range, $f(4)$, $f(0)$ and $f(-6)$.

*Solution: Domain: all reals, range $\{5, 2, -3\}$, $f(4) = 5$, $f(0) = 2$ and $f(-6) = -3$*

**Activity:**

- Use the Bellringer to review graphing piecewise functions, domain, range, and function notation for particular domains. Ask students what kinds of lines each of the pieces are and to find the slope of each of the lines. Describe step functions as a series of horizontal lines.

- Ask the students for examples in real life of step functions (e.g., shoe sizes, postage rates, tax brackets).

- **Greatest Integer Functions**
  
  o On the graphing calculator, have the students graph $y = \text{Int}(x)$. If it looks like the first graph, the calculator is in connected mode. Have students change the mode to dot mode.
• Have the students write a piecewise function for the graph above and state the domain and range.

\[
\begin{align*}
0 & \text{ if } 0 \leq x < 1 \\
-1 & \text{ if } -1 \leq x < 0 \\
-2 & \text{ if } -2 \leq x < -1 \\
-3 & \text{ if } -3 \leq x < -2 \\
3 & \text{ if } 3 \leq x < 4 \\
2 & \text{ if } 2 \leq x < 3 \\
1 & \text{ if } 1 \leq x < 2
\end{align*}
\]

\[f(x) = \begin{cases} 
3 & \text{if } 3 \leq x < 4 \\
2 & \text{if } 2 \leq x < 3 \\
1 & \text{if } 1 \leq x < 2 \\
0 & \text{if } 0 \leq x < 1 \\
-1 & \text{if } -1 \leq x < 0 \\
-2 & \text{if } -2 \leq x < -1 \\
-3 & \text{if } -3 \leq x < -2
\end{cases}
\]

Domain: all reals, Range: integers

• Define the above piecewise function symbolically as \( f(x) = \lfloor x \rfloor \) and verbally as “the greatest integer less than or equal to \( x \)” or, in other words, a “round down” function. The graph is said to have “jump discontinuities” at the integers.

o Have students find the following: (1) \( 7.1 \) (2) \( 1.8 \) (3) \( \pi \) (4) \( -6.8 \) (5) \( -2.1 \)

Solution: (1) 7, (2) 1, (3) 3, (4) –7, (5) –2

• Have students solve the following equations and write the answers in set notation:

(1) \[ \frac{2x}{7} = 1 \] 

Solutions:

(1) \( \frac{7}{2} \leq x < 7 \) 

(2) \[ \left\lfloor 3x \right\rfloor = 12 \] 

Solutions:

(2) \( 4 \leq x < \frac{13}{3} \)

• Translating Graphs of Greatest Integer Functions

Put students in pairs to work the following problem:

Using what you learned about the translations of \( y = a|x - h| + k \), graph the following by hand and check on your calculator:

(1) \( f(x) = \left\lceil x \right\rceil + 2 \)

\[ g(x) = \left\lceil x \right\rceil + 2 \].

What is the difference in these graphs?

(2) \( f(x) = 2\left\lceil x \right\rceil \)

\[ g(x) = 2\left\lceil x \right\rceil \].

What is the difference in these graphs?

(3) \( f(x) = -\left\lfloor x \right\rfloor \)

\[ g(x) = \left\lfloor x \right\rfloor \].

What is the difference in these graphs?
Critical Thinking Writing Activity

Prior to September, 1998, taxi fares from Washington DC to Maryland were described as follows: 1st increment $2.00 up to ½ mile
Additional increments $0.70 for each ½ mile

Graph the fares and write the piecewise function for 0 to 4 miles. Discuss why this is a step function and how this graph is different from the greatest integer function?

Solution: This graph is different from greatest integer because the range is \{0, 2, 2.70, 3.40, ..., 2 + .70k, k \in \mathbb{Z}\} and jump discontinuities occur at every increment of ½ instead of 1. The closed dot is on the right side of the horizontal step instead of on the left.

Step Function Data Research project

Have students find data on the Internet or in the newspaper that is conducive to creating a Step Function Graph. Instruct them to graph the function on ½ sheet of poster paper, decorate it relative to the topic, write the equation, domain and range and use it to interpolate and extrapolate an answer a real world question.

Activity 9: Compositions of Functions (GLEs: 1, 4, 8, 10, 24, 25)

The students will combine functions to create new functions, decompose functions into simpler functions, and find their domains and ranges.

Bellringer: \( f(x) = -2x + 5 \) Find the following:

1. \( f(0) \)
2. \( f(-2) \)
3. \( f(a) \)
4. \( f(a + 1) \)

Solutions: (1) 5, (2) 1, (3) \(-2a + 5\), (4) \(-2a + 3\)

Activity:

- Use the Bellringer to check for understanding of function notation. In problem 4, students replaced the variable \( x \) with an algebraic expression, \( a + 1 \), and created a new function. This is called composition of functions. Provide this definition of composite function to students: Given two functions \( f(x) \) and \( g(x) \), the composite function, \( f(g(x)) \) or \((f \circ g)(x)\), is the operation of applying \( g(x) \) to the inputted \( x \) values and then \( f(x) \) to the output of \( g(x) \).

- Give the example \( f(x) = -2x + 5 \) and \( g(x) = x + 1 \). Find \( f(g(-2)) \) and \( g(f(-2)) \) and demonstrate with the function machine.

\[
\begin{array}{ccc}
\text{x = -2} & \text{g(x) = 3x + 1} & -5 \\
\text{f(x) = -2x + 5} & 15 \\
\text{x = -2} & \text{f(x) = -2x + 5} & 9 \\
\text{g(x) = 3x + 1} & 28
\end{array}
\]
• Calculating the values of compositions without a rule
Use the table to find the following:

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>f(x)</th>
<th>g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>9</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>3</td>
<td>7</td>
</tr>
</tbody>
</table>

Solution: (1) 3, (2) 12, (3) 9, (4) cannot be determined from this table

• Using \( f(x) = -2x + 5 \) and \( g(x) = x + 1 \), have students find a rule for \( f(g(x)) \) and \( g(f(x)) \).

Solution:

\[
\begin{array}{c|c|c|c}
\text{x} & \text{g(x)} & \text{3x+1} & \text{f(x)} & \text{= -2x + 5} & \text{= -2(3x+1) + 5} & \text{= -6x + 3} \\
\hline
\text{x} & \text{f(x)} & \text{= -2x + 5} & \text{g(x)} & \text{= 3x + 1} & \text{= 3(-2x +5) + 1} & \text{= -6x +16} \\
\end{array}
\]

• Support the composition of functions with a graphing calculator
In order to graph the composition \( f(g(x)) \) on a graphing calculator, have students enter \( g(x) \) into \( y_1 = 3x + 1 \) and turn it off so it will not graph. Next, ask students to enter \( f(x) \) into \( y_2 \) as follows \( y_2 = -2(y_1) + 5 \) and graph. Then have students graph the answer from the function machine, \( y_3 = -6x + 3 \), to see if they are the same graph.

• Have students practice with polynomial functions: \( f(x) = 2x + 1 \) and \( g(x) = 4x^2 + 3 \). Ask them to find \( f(g(x)) \) and \( g(f(x)) \) and check on the calculator.

Solution: \( f(g(x)) = 8x^2 + 7 \), \( g(f(x)) = 16x^2 + 16x + 7 \)

• One of the most difficult compositions that is also very necessary for higher mathematics is finding \( \frac{f(x+h)-f(x)}{h} \). Give the students the following to build this concept:

\( f(x) = 3x^2 + 2 \). Ask students to find the following:

(1) \( f(3) \)
(2) \( f(a) \)
(3) \( f(a + b) \)
(4) \( f(x) + h \)
(5) \( f(x+h) \)
(6) \( \frac{f(x+h)-f(x)}{h} \)

Solutions:

(1) 29
(2) \( 3a^2 + 2 \)
(3) \( 3a^2 + 6ab + 3b^2 + 2 \)
• Most functions are compositions of functions. Have students work backwards to determine the base functions. Then have students complete the following chart.

<table>
<thead>
<tr>
<th></th>
<th>( f(g(x)) )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((x + 4)^2 + 5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( \sqrt{x - 4} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>((4x - 1)^2)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>([x + 2])</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>([x - 2]^2 + 4)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Solutions:
\((1) f(x) = x^2 + 5 \quad g(x) = x + 4\)
\((2) f(x) = \sqrt{x} \quad g(x) = x - 4\)
\((3) f(x) = x^2 \quad g(x) = 4x - 1\)
\((4) f(x) = |x| \quad g(x) = x + 2\)
\((5) f(x) = [x]^2 + 4 \quad g(x) = x - 2\)

• Domains and ranges of compositions
Have students fill in the table below:

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>Domain of ( f(x) )</th>
<th>Domain of ( g(x) )</th>
<th>( f(g(x)) )</th>
<th>Domain of ( f(g(x)) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \sqrt{x - 3} )</td>
<td>( x + 1 )</td>
<td></td>
<td>( x + 1 )</td>
<td>( \sqrt{x - 3} )</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>( x + 1 )</td>
<td>( \sqrt{x - 3} )</td>
<td></td>
<td>( \sqrt{x - 3} )</td>
<td>( x + 1 )</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{x} )</td>
<td>( 2x + 4 )</td>
<td></td>
<td>( 2x + 4 )</td>
<td>( \frac{1}{x} )</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( 2x + 4 )</td>
<td>( \frac{1}{x} )</td>
<td></td>
<td>( \frac{1}{x} )</td>
<td>( 2x + 4 )</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( \sqrt{x} )</td>
<td>( x^2 )</td>
<td></td>
<td>( x^2 )</td>
<td>( \sqrt{x} )</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( x^2 )</td>
<td>( \sqrt{x} )</td>
<td></td>
<td>( \sqrt{x} )</td>
<td>( x^2 )</td>
<td></td>
</tr>
</tbody>
</table>

Solutions:
\((1) x \geq 3 \quad all \ reals \quad \sqrt{x - 2} \quad x \geq 2\)
\((2) all \ reals \quad x \geq 3 \quad \sqrt{x - 3 + 1} \quad x \geq 3\)
\((3) x \neq 0 \quad all \ reals \quad \frac{1}{2x + 4} \quad x \neq -2\)
\((4) all \ reals \quad x \neq 0 \quad \frac{2}{x} \quad x \neq 0\)
\((5) x \geq 0 \quad all \ reals \quad \sqrt{x^2} \quad all \ reals\)
\((6) all \ reals \quad x \geq 0 \quad \left(\sqrt{x}\right)^2 \quad x \geq 0\)
• From the chart above, have students develop the rule for determining the **domain of a composition**: To determine the domain of the composition \( f(g(x)) \), find the domain of \( g(x) \) and further restrict it for the composition \( f(g(x)) \). Note that the domain restrictions on \( f(x) \) have no consequences on the composition.

• **Critical Thinking Writing Activity**

  (1) The price the store pays for a CD is determined by the function \( f(x) = x + 3 \), where \( x \) is the wholesale price. The price the store charges for the CD is determined by the function \( g(x) = 2x + 4 \) where \( x \) is the price the store pays. How can this be expressed as a composition of functions? Find the price to the customer if the wholesale price of the CD sells is $12.

  (2) Explain the difference in the way you compute \( f(a + h) \) and \( f(a) + h \) and verbally work through the steps to compute both for the function \( f(x) = 4x^2 - 1 \).

**Solutions:**

(1) \( g(f(x)) = 2(x + 3) + 4, \) \( g(f(12)) = 34 \)

(2) In \( f(a + h) \) you add \( h \) to \( a \) then substitute it in the function. In \( f(a) + h \), you add \( h \) after you have substituted \( a \) into the function.

\[
\begin{align*}
  f(a + h) &= 4a^2 - 8ah + 4h^2 - 1, \\
  f(a) + h &= 4a^2 - 1 + h
\end{align*}
\]

**Activity 10: Inverse Functions (GLEs: 1, 4, 6, 8, 10, 16, 25, 28, 29)**

The student will find inverse relations and determine whether that relation is a function. They will also determine the domain and range of the inverse function and see how the graphs of a function and its inverse are related.

**Bellringer:**

(1) Graph \( f(x) = 2x + 4 \) and \( g(x) = \frac{1}{2}x - 2 \) on the same graph.

(2) Find \( f(3) \) and \( g(10) \).

(3) What point do they share?

(4) Find \( f(g(x)) \) and \( g(f(x)) \).

**Solutions:**

(2) \( f(3) = 10 \) and \( g(10) = 3 \)

(3) They share \((-4, -4)\)

(4) \( f(g(x)) = g(f(x)) = x \)

**Activity:**

• Use the Bellringer to review graphing lines, using function notation, and composing functions. Have students graph both equations on their calculators and use the feature, ZOOM square. Note how the \( x \) and \( y \) values have swapped. Ask, Are both graphs functions? Have students graph \( y = x \) on the same graph and make comparisons.

• Define **inverse relation** as any relation that swaps the independent and dependent variables. Have students determine whether the inverse of every function is a function.
Have students use problems (given below) from the Bellringer in Activity 1, swap the $x$ and $y$ in each relation, and determine if the new relation is a function and why.

1. the set of ordered pairs \{\((x, y) : (1, 2), (3, 5), (3, 6), (7, 5), (8, 2)\}\}
2. the set of ordered pairs \{\((s, d) : (1, 2), (2, 4), (3, 9), (−1, 1), (−2, 4), (−3, 9)\}\}
3. the relationship “$x$ is a student of $y$”
4. the relationship “$x$ is the biological daughter of mother $y$”
5. the equation $2x + 3y = 6$
6. the equation $x + y^2 = 9$
7. the equation $y = x^2 + 4$
8. the graph of the circle $x^2 + y^2 = 9$

Answer the following questions:

9. How can you look at a set of ordered pairs and determine if the inverse relation will be a function?
10. How can you look at an equation and determine if the inverse relation will be a function?
11. How can you look at a graph and determine if the inverse relation will be functions?
12. How can you look at a verbal statement and determine if the inverse verbal relationship will be a function?
13. Draw a function machine for $f(g(x))$ and $g(f(x))$ using the functions in the Bellringer ($f(x) = 2x + 4$ and $g(x) = \frac{1}{2}x − 2$) to find $f(g(2))$ and $g(f(2))$.

Solution:

1. New relation: \{\((y, x) : (2, 1), (5, 3), (6, 3), (5, 7), (2, 8)\}\} not function – the number 2 maps onto 1 and 8
2. New relation:\{\((d, s) : (1, 1), (4, 2), (9, 3), (1, −1), (4, −2), (9, −3)\}\}, not function - dependent variables are repeated
3. New relation: “$y$ is a teacher of $x$” not function - a teacher can have more than one student
4. New relation: “$y$ is the biological mother of daughter $x$” not function – a mother can have many daughters
5. New relation: $2y + 3x = 6$, yes
6. New relation: $y + x^2 = 9$, yes
7. New relation: $x = y^2 + 9$, not function – two $y$ values for each $x$
8. New relation: same as the old relation, not function - two $y$ values for each $x$
9. Looking at ordered pairs, neither the independent nor the dependent variables can be repeated
10. Looking at the equation, neither the $x$ nor the $y$ can be raised to an even power
11. Looking at the verbal statement, neither the independent nor the dependent variable may be repeated
12. Looking at the graph, neither a vertical nor a horizontal line can intersect the graph at two points.
13. 

\[
\begin{align*}
x = 2 & \quad f(x) = 2x + 4 & \quad 8 & \quad g(x) = \frac{1}{2}x - 2 & \quad 2 \\
x = 2 & \quad g(x) = \frac{1}{2}x - 2 & \quad -1 & \quad f(x) = 2x + 4 & \quad 2 \\
\end{align*}
\]
• Have students define inverse function in the following ways:
  o **Symbolically**
    If \( f \) is a function, then \( f^{-1}(x) \) is the inverse function if \( f(f^{-1}(x)) = f^{-1}(f(x)) = x \).
  o **Verbally**
    When you compose a function with its inverse, the resulting rule is the identity function \( y = x \).
  o **Numerically**
    The domain of a function is the range of the inverse function and vice versa.
  o **Process for finding symbolic inverse functions**
    Replace the \( f(x) \) with \( y \). Interchange \( x \) and \( y \) and solve for \( y \). Rename \( y \) as \( f^{-1}(x) \).

• Find the inverse of the following functions, graph \( f(x) \) and \( f^{-1}(x) \), and find the domain and range of both.
  (1) \( f(x) = 3x + 4 \)
  (2) \( f(x) = 2|x - 1| \) on the domain \( x \leq 1 \)
  (3) \( f(x) = \sqrt{x} \)

**Solution:**
(1) \( f^{-1}(x) = \frac{1}{3}x - \frac{4}{3} \), Domain and range all reals

(2) \( f^{-1}(x) = -x + 2 \) on the domain \( x \geq 0 \),

Domain \( f(x) = (-\infty, 1] \), range \( f(x) = [0, \infty) \),

Domain \( f^{-1}(x) = [0, \infty) \), range \( f^{-1}(x) = (-\infty, 1] \)
(3) \( f^{-1}(x) = x^2 \) on the domain \( x \geq 0 \).

Domain \( f(x) = [0, \infty) \), range \( f(x) = [0, \infty) \),

Domain \( f^{-1}(x) = [0, \infty) \), range \( f^{-1}(x) = [0, \infty) \)

Critical Thinking Writing Activity
The temperature \( T \), in degrees Fahrenheit, of a cold potato placed in a hot oven is given by \( T = f(t) \), where \( t \) is the time in minutes since the potato was put in the oven. What is the practical meaning of the symbolic statement, \( f(20) = 100 \)? Discuss the practical meaning of the statement \( f^{-1}(4) = 10 \) (use units in your sentence.)

Solution: \( f(20) = 2 \) means that after 20 minutes in the oven, the potato has risen to 100°F. \( f^{-1}(120) = 25 \) means that if the temperature is 120°F, then the potato must have been in the oven for 25 minutes.

Sample Assessments

General Assessments

- The teacher will use Bellringers as ongoing informal assessments.
- The teacher will collect the Little Black Book of Algebra II Properties and grade for completeness at the end of the unit.
- The teacher will monitor student progress using small quizzes to check for understanding during the unit on such topics as the following:
  1. speed in graphing lines
  2. changing among set notation, interval notation, absolute value notation, and number lines.
  3. graphing piecewise linear functions
  4. graphing absolute value functions
- The student will demonstrate proficiency on two comprehensive assessments:
  1. functions, graphing piecewise linear functions, stating solution sets in all notations
  2. graphing absolute value functions and greatest integer functions, changing absolute value functions to piecewise functions, finding compositions of functions and inverse functions
Activity-Specific Assessments

- **Activities 1, 3 and 5:** The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the following rubric:

  *Grading Rubric for Critical Thinking Writing Activities:*
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols
  3 pts./graph - correct graphs (if applicable)
  3 pts./solution - correct equations, showing work, correct answer
  3 pts./discussion - correct conclusion

- **Activity 5:** The teacher will evaluate the Translating Graphs of Lines – Discovery Worksheet (see activity) using the following rubric:

  *Grading Rubric for Discovery Worksheets*
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols
  2 pts./graph - correct graphs and equations (if applicable)
  5 pts/discussion - correct conclusions

- **Activity 7:** The teacher will evaluate the Translating Absolute Value Function Graphs in the form $f(x) = a|x - h| + k$ - Discovery Worksheet (see activity) by using the Grading Rubric for Discovery Worksheets in Activity 5. The teacher will evaluate the 2nd Critical Thinking Writing Activity (see activity) by using grading rubric provided in the assessment for Activity 1.

- **Activity 8:** The teacher will evaluate the Translating Graphs of Greatest Integer Functions – Discovery Worksheet (see activity) by using the Grading Rubric for Discovery Worksheets in Activity 5. The teacher will evaluate the Critical Thinking Writing Activity (see activity) by using rubric in Activity 1. The student will complete the Step Function Data Research project and the teacher will evaluate the project using the following:

  *Grading Rubric for Data Research:*
  10 pts. - data with proper documentation
  10 pts. - graph
  10 pts. - equations, domain, range,
  10 pts. - real world problem using interpolation and extrapolation, with correct answer
  10 pts. - poster - neatness, completeness, readability

- **Activity 9:** The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the rubric provided in the assessment for Activity 1.
• **Activity 10:** The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the rubric provided in the assessment for Activity 1.
Time Frame: Approximately four weeks

Unit Description

This unit develops the procedures for factoring polynomial expressions in order to solve polynomial equations and inequalities and introduces the graphs of polynomial functions using technology.

Student Understandings

Even in this day of calculator solutions, symbolically manipulating algebraic expressions is still an integral skill for students to advance to higher mathematics. However, these operations should be tied to real-world applications so students understand the relevance of the skills. Students need to understand the reasons for factoring a polynomial and determine the correct strategy to use. They should understand the relationship of the zero–product property to the solutions of polynomial equations and inequalities and connect these concepts to the zeroes of a graph of a polynomial function.

Guiding Questions

1. Can students use the rules of exponents to multiply and divide monomials and numbers in scientific notation?
2. Can students add and subtract polynomials and apply to geometric problems?
3. Can students multiply polynomials and identify special products?
4. Can students expand a binomial using Pascal’s triangle?
5. Can students factor expressions using the greatest common factor and factor binomials containing the difference in two perfect squares and the sum and difference in two perfect cubes?
6. Can students factor perfect square trinomials and general trinomials?
7. Can students factor polynomials by grouping?
8. Can students select the appropriate technique for factoring?
9. Can students apply multiplication of polynomials and factoring to geometric problems?
10. Can students factor in order to solve polynomial equations using the zero–product property?
11. Can students relate factoring a polynomial to the zeroes of the graph of a polynomial?
12. Can students relate multiplicity to the effects on the graph of a polynomial?
13. Can students determine the effects on the graph of factoring out a greatest common constant factor?
14. Can students predict the end behavior of a polynomial based on the degree and sign of the leading coefficient?
15. Can students sketch a graph of a polynomial in factored form using end–behavior and zeroes?
16. Can students solve polynomial inequalities by the factor/sign chart method?
17. Can students solve polynomial inequalities by examining the graph of a polynomial using technology?

### Unit 2 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Number and Number Relations</strong></td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>Read, write, and perform basic operations on complex numbers (N-1-H) (N-5-H)</td>
</tr>
<tr>
<td>2.</td>
<td>Evaluate and perform basic operations on expressions containing rational exponents (N-2-H)</td>
</tr>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>5.</td>
<td>Factor simple quadratic expressions including general trinomials, perfect squares, difference of two squares, and polynomials with common factors (A-2-H)</td>
</tr>
<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
</tr>
<tr>
<td>7.</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)</td>
</tr>
<tr>
<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H)(P-5-H)</td>
</tr>
<tr>
<td>9.</td>
<td>Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-3-H)</td>
</tr>
<tr>
<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
<tr>
<td>19.</td>
<td>Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>22.</td>
<td>Explain the limitations of predictions based on organized sample sets of data (D-7-H)</td>
</tr>
<tr>
<td>24.</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
</tbody>
</table>
Sample Activities

Ongoing: Little Black Book of Algebra II Properties - Polynomial Equations and Inequalities

The following is a list of properties to be included in the Little Black Book of Algebra II Properties. Add other items as appropriate.

2.1 Laws of Exponents - record the rules for adding, subtracting, multiplying and dividing quantities containing exponents, raising an exponent to a power, and using zero and negative values for exponents.

2.2 Polynomial Terminology – define and write examples of monomials, binomials, trinomials, polynomials, the degree of a polynomial, a leading coefficient, a quadratic trinomial, a quadratic term, a linear term, a constant, and a prime polynomial.

2.3 Special Binomial Products – define and give examples of perfect square trinomials and conjugates, write the formulas and the verbal rules for expanding the special products \((a + b)^2\), \((a – b)^2\), \((a + b)(a – b)\), and explain the meaning of the acronym, FOIL.

2.4 Binomial Expansion using Pascal’s Triangle – create Pascal’s triangle through row 7, describe how to make it, explain the triangle’s use in binomial expansion, and use the process to expand both \((a + b)^5\) and \((a – b)^5\).

2.5 Common Factoring Patterns - define and give examples of factoring using the greatest common factor of the terms, the difference in two perfect squares, the sum/difference in two perfect cubes, the square of a sum/difference \((a^2 + 2ab + b^2, a^2 – 2ab + b^2)\), and the technique of grouping.

2.6 Zero–Product Property – explain the zero–product property and its relevance to factoring: Why there is a zero–product property and not a property like it for other numbers?

2.7 Solving Polynomial Equations – identify the steps in solving polynomial equations, define double root, triple root, and multiplicity, and provide one reason for the prohibition of dividing both sides of an equation by a variable.

2.8 Introduction to Graphs of Polynomial Functions – explain the difference between roots and zeroes, define end behavior of a function, indicate the effect of the degree of the polynomial on its graph, explain the effect of the sign of the leading coefficient on the graph of a polynomial, and describe the effect of even and odd multiplicity on a graph.
2.9 **Polynomial Regression Equations** – explain the Method of Finite Differences to determine the degree of the polynomial that is represented by data.

2.10 **Solving Polynomial Inequalities** – indicate various ways of solving polynomial inequalities such as using the sign chart and using the graph. Provide two reasons for the prohibition against dividing both sides of an inequality by a variable.

**Activity 1: Multiplying Binomials and Trinomials (GLEs: 1, 2, 19)**

The students will apply the simple operations of polynomials learned in Algebra I to multiply complex polynomials.

**Bellringer:** Simplify the following expression: \((x^2)^3 + 4x^2 - 6x^3(x^5 - 2x) + (3x^4)^2 + (x + 3)(x - 6)\)

*Solution:* \(3x^8 + x^6 + 12x^4 + 5x^2 - 3x - 18\)

**Activity:** Divide the students into groups of three to check their answers to the Bellringer and have them write three rules they used. Put these rules on large sheets of paper and tape them to the board to compare with other groups. In addition to the rules of exponents, look for the commutative, associative, and distributive properties, FOIL, combining like terms, and arranging the terms in descending order. Review the definitions of monomial, binomial, trinomial, polynomial, degree of polynomial, and leading coefficient. Have each group expand \((a + b)^2\), \((a - b)^2\), and \((a + b)(a - b)\) and write the words for finding these special products, again comparing answers with other groups and voting on the best verbal explanation. Define the word conjugate. After expanding several binomial and trinomial products, have students work the following application problems:

1. The length of the side of a square is \(x + 3\) cm. Express the perimeter and area as a polynomial function using function notation.
2. A rectangular box is \(2x + 3\) feet long, \(x + 1\) feet wide and \(x - 2\) feet high. Express the volume as a polynomial in function notation.
3. For the following figures, write an equation showing that the area of the large rectangle is equal to the sum of the areas of the smaller rectangles.

*Solution:*

1. \(p(x) = 4x + 12\ cm, A(x) = x^2 + 6x + 9\ cm^2\)
2. \(V(x) = 2x^3 + x^2 - 7x - 6\)
3A. \((x + 2)(x + 1) = x^2 + lx + 1x + 1x + 1 + 1 = x^2 + 3x + 2\)
3B. \((2x + 1)(x + 2) = x^2 + x^2 + lx + 1x + 1x + 1x + 1x + 1 + 1 = 2x^2 + 5x + 2\)
Activity 2: Using Pascal’s Triangle to Expand Binomials (GLEs: 1, 2, 8, 27)

The focus of this activity is to find a pattern in coefficients in order to quickly expand a binomial using Pascal’s Triangle and to use the calculator \( _nC_r \) button to generate Pascal’s triangle.

**Bellringer:** Expand the following binomials:

1. \((a + b)^0\)
2. \((a + b)^1\)
3. \((a + b)^2\)
4. \((a + b)^3\)
5. \((a + b)^4\)

**Solutions:**

1. 1
2. \(a + b\)
3. \(a^2 + 2ab + b^2\)
4. \(a^3 + 3a^2b + 3ab^2 + b^3\)
5. \(a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\)

**Activity:**

- Have five of the students each work one of the Bellringer problems on transparencies while the rest of the students work in their notebooks. Have students compare answers to check for understanding of the FOIL process.

- Write the coefficients in triangular form and have students find a pattern and write a rule to develop Pascal’s triangle and determine which row is used in which expansion. Have students expand \((a – b)^2\) and \((a – b)^3\) by hand and modify the rule for differences.

- Have students expand \((a + b)^6\), \((a – b)^7\), and \((2x + 3y)^4\) using Pascal’s triangle and then simplify each.

- Lead a short discussion of combinations and how many subset combinations there would be of \(\{3, 5, 9\}\) if taken two at a time. Discuss the symbols \( _nC_r \), \( _nC_{10} \), and review the concept of combinations introduced in Algebra I.

- Have the students use the set \(\{a, b, c, d\}\) and list the sets which represent 4 elements taken 1 at a time or \(_4C_1\), 4 elements taken 2 at a time or \(_4C_2\), 4 elements taken 3 at a time or \(_4C_3\), and 4 elements taken 4 at a time or \(_4C_4\), and compare these answers to Pascal’s triangle.

- Locate the \( _nC_r \) button on the graphing calculator and use it to find the next row on Pascal’s triangle. Enter \(y = 7_{10}C_x\) in the calculator. Set the table to start at 0 with increments of 1. Create the table and compare the values to Pascal’s Triangle. Use this feature to expand \((a + b)^9\).
Activity 3: Factoring Polynomials (GLEs: 1, 2, 5, 10, 24, 27)

In this activity, students will factor a polynomial containing common factors, a perfect square trinomial, and binomials that are the difference of two perfect squares or the sum or difference of two perfect cubes.

Bellringer: Find the greatest common factor:
1. 24, 36, 60
2. \(8x^2y^3, 12x^3y, 20x^2y^2\)

<table>
<thead>
<tr>
<th>Solution</th>
<th>(1) 12</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(2) 4x^2y</td>
</tr>
</tbody>
</table>

Activity:
- Use the Bellringer to review the definition of factor and discuss the greatest common factors (GCF) of numbers and monomials. Have students factor common factors out of several polynomials.
- Have students examine the following first three trinomials and use the verbal rules written in Activity 1 to determine how to rewrite the trinomials in factored form. Then have them apply the rules to more complicated trinomials (problems 4 and 5).

\[
\begin{align*}
1. & \quad a^2 + 6a + 9 \\
2. & \quad s^2 - 8s + 16 \\
3. & \quad 16h^2 - 25 \\
4. & \quad 9x^2 + 42x + 49 \\
5. & \quad 64x^2 - 16xy + y^2
\end{align*}
\]

<table>
<thead>
<tr>
<th>Solution</th>
<th>(1) ((a + 3)^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(2) ((s - 4)^2)</td>
</tr>
<tr>
<td></td>
<td>(3) ((4h - 5)(4h + 5))</td>
</tr>
<tr>
<td></td>
<td>(4) ((3x + 7)^2)</td>
</tr>
<tr>
<td></td>
<td>(5) ((8x - y)^3)</td>
</tr>
</tbody>
</table>

- Have students expand the following \((a + b)(a^2 - ab + b^2)\) and \((a - b)(a^2 + ab - b^2)\) and write two verbal rules that will help them factor \(a^3 - b^3\) and \(a^3 + b^3\).

- Application problems:
  1. The area of a rectangle can be represented by \(25x^2 - 16\). What is a binomial expression for each side?
  2. A small square of plastic is to be cut from a square plastic box cover. Express the area of the shaded form in factored form and show that it is equal to the area of the shaded region in the second figure.

\[
\begin{align*}
(1) \quad & \quad (5x-4)(5x+4) \\
(2) \quad & \quad x^2 - y^2 = (x + y)(x - y)
\end{align*}
\]
Activity 4: Factoring Quadratic Trinomials (GLEs: 1, 2, 5, 8, 9, 10, 29)

Students expand and factor to find the relationships necessary to factor quadratic trinomials.

Bellringer: Expand the following:

<p>| | |</p>
<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(x + 4)(x + 3)</td>
</tr>
<tr>
<td>(2)</td>
<td>(x – 4)(x – 3)</td>
</tr>
<tr>
<td>(3)</td>
<td>(x + 4)(x – 3)</td>
</tr>
<tr>
<td>(4)</td>
<td>(x – 4)(x + 3)</td>
</tr>
<tr>
<td>Solution:</td>
<td></td>
</tr>
<tr>
<td>(1)</td>
<td>x^2 + 7x + 12</td>
</tr>
<tr>
<td>(2)</td>
<td>x^2 – 7x + 12</td>
</tr>
<tr>
<td>(3)</td>
<td>x^2 + x – 12</td>
</tr>
<tr>
<td>(4)</td>
<td>x^2 – x – 12</td>
</tr>
</tbody>
</table>

Activity:
Use the Bellringer to discuss the relationships between the middle term as being the sum of the inner and outer terms and the last term as the product of the last terms. Discuss the signs and have students write a rule. Have students factor numerous trinomials.

- **Application:**
  1. A rectangular window has an area (x^2 + 8x + 15) m^2. Find the factors that represent the sides of the window.
  2. The area of a rectangular lot is (5x^2 – 3x – 2) ft^2. What is the perimeter of the lot?
  3. Write a quadratic trinomial that can be used to find the side of the square if the area less the side is twenty. (Hint: Isolate zero and factor in order to find the possible length of the side.)

  Solution:
  
<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(x + 5)(x+3)</td>
</tr>
<tr>
<td>(2)</td>
<td>6x+1</td>
</tr>
<tr>
<td>(3)</td>
<td>s^2 – s = 20 ⇒ s^2 – s – 20 = 0 ⇒ (s – 5)(s + 4) = 0 ⇒ s = 5 or s = –1</td>
</tr>
</tbody>
</table>

  A side must be positive; therefore, s = 5.

Activity 5: Factoring by Grouping (GLEs: 1, 2, 5, 8, 9, 10, 29)

Students will review all methods of factoring and factor a polynomial of four or more terms by grouping terms.

Bellringer: Factor completely and explain which special product you used:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>2x^2y^3 + 6xy^2 + 8x^3y</td>
</tr>
<tr>
<td>(2)</td>
<td>4x^2 + 4x + 1</td>
</tr>
<tr>
<td>(3)</td>
<td>16x^2 – 36y^2</td>
</tr>
<tr>
<td>(4)</td>
<td>1 + 8x^3</td>
</tr>
<tr>
<td>(5)</td>
<td>9x^2 – 12x + 4</td>
</tr>
<tr>
<td>(6)</td>
<td>3x^2 + 6x</td>
</tr>
<tr>
<td>Solutions:</td>
<td></td>
</tr>
<tr>
<td>(1)</td>
<td>2xy(xy^2 + 3y + 4x^2)</td>
</tr>
<tr>
<td>(2)</td>
<td>(2x + 1)^2</td>
</tr>
<tr>
<td>(3)</td>
<td>(4x – 6y)(4x + 6y)</td>
</tr>
<tr>
<td>(4)</td>
<td>(1 + 2x)(1 – 2x + 4x^2)</td>
</tr>
<tr>
<td>(5)</td>
<td>(3x – 2)^2</td>
</tr>
<tr>
<td>(6)</td>
<td>3x(x + 2)</td>
</tr>
</tbody>
</table>

Activity:
- Have the students use the Bell ringer to develop the steps for factoring a polynomial completely – (1) factor out GCF, (2) if the polynomial is a binomial, look for special
products such as difference of two perfect square or the sum/difference of two perfect cubes, and (3) if the polynomial is a trinomial, look for a perfect square binomial or FOIL.

- Give the students a polynomial made of four monomials such as $8x^3 + 4x^2 - x^3 - x^2$. Allow them to work in pairs to brainstorm possible methods of factoring and possible ways to group two monomials in order to apply one of the basic factoring patterns. Develop factoring by grouping and add to the list. Provide students with guided practice problems.

- **Application:**
  (1) The area of a rectangle is $xy + 2y + x + 2$ ft$^2$. Find the possible lengths of the sides.
  (2) Prove that the ratio of the area of the circular shaded region below to the rectangular shaded region equals $\pi$.

![Diagram]

**Solution:**
(1) $(y + 10)$ and $(x + 2)$
(2) $\text{Area}_1 = \pi (R^2 - r^2)$ and $\text{Area}_2 = (R - r)(R + r)$

$$\frac{\text{Area}_1}{\text{Area}_2} = \pi$$

**Activity 6: Solving Equations by Factoring (GLEs: 1, 2, 5, 6, 8, 9, 24, 29)**

In this activity, the students will develop the zero–product property and use it and their factoring skills to solve polynomial equations.

**Bellringer:**
Solve: (1) $2x = 16$    (2) $2x^2 = 16x$

**Solution:** (1) $x = 8$ , (2) $x = 8$ and $x = 0$

**Activity:**
- Determine how many students got both answers in Bellringer problem 2 and use this to start a discussion about division by a variable. *(Do not divide both sides of an equation by a variable because the variable may be zero.)* Define division as $\frac{a}{b} = c$ if and only if $bc = a$ and have students explain why division by zero is “undefined.”
• Have a student who worked problem 2 correctly show his/her work on the board. *(He/she should have isolated zero and factored.)* Have the students develop the Zero–Product Property. Make sure students substitute to check their answers. Review the use of *and* and *or* in determining the solution sets in compound sentences. Compare the solution for problem 2 to someone who solved the problem \(x(x + 2) = 8\) incorrectly as \(\{8, 6\}\). Have students substitute solutions to check answers and discuss why there is no “Eight Property.” Use guided practice to allow students to solve several more quadratic polynomial equations using factoring.

• Have the students solve the following and discuss double and triple roots and multiplicity:
  1. \((x - 4)(x - 3)(x + 2) = 0\)
  2. \(y^3 - 3y^2 = 10y\)
  3. \(x^2 + 6x = -9\)
  4. \((x^2 + 4x + 4)(x + 2) = 0\)

  *Solution:*
  1. \(\{-2, 3, 4\}\),
  2. \(\{0, 5, -2\}\),
  3. \(\{-3\}\),
  4. \(\{-2\}\)

• Have students develop the steps for solving an equation by factoring:
  1. Write in Standard Form (Isolate zero)
  2. Factor
  3. Use the zero–product property
  4. Find the solutions
  5. Check

• **Application Problems**
  Divide the students in groups to set up and solve these application problems:
  1. The perimeter of a rectangle is 50 in. and the area is 144 in\(^2\). Find the dimensions of the rectangle.
  2. A concrete walk of uniform width surrounds a rectangular swimming pool. Let \(x\) represent this width. If the pool is 6 ft. by 10 ft. and the total area of the pool and walk is 96 ft\(^2\). Find the width of the walk.
  3. The longer leg of a right triangle has a length 1 in. less than twice the shorter leg. The hypotenuse has a length 1 in. greater than the shorter leg. Find the length of the three sides of the triangle.

  *Solutions:*
  1. 16 in. by 9 in, 2 ft.
  2. \(\frac{1}{2}\) foot
  3. 2.5 in., 2 in., and 1.5 in.

**Activity 7: Investigating Graphs of Polynomial Functions (GLEs: 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 19, 25, 27, 28)**

In this activity, students will use technology to graph polynomial functions to find the relationship between factoring and finding zeroes of the function. They will also discover end–behavior and the effects of a common constant factor, even and odd degrees, and the sign of the leading coefficient on the graph of a function.
Bellringer:
(1) Factor \(0 = x^3 + 4x^2 + 3x\) and solve.
(2) Graph \(y = x^3 + 4x^2 + 3x\) on your calculator and find the zeroes.
\[(\text{Solution: } \{0, -3, -1\})\]

Activity:
- Use the Bellringer to review calculator skills for finding zeroes, adjusting the window to show a comprehensive graph displaying both intercepts and the maximum and minimum points. Have students determine why the solutions to the equation are the zeroes of the graph. Discuss the end–behavior.
- Have the students graph \(f(x) = x^3 - 3x^2 - 10x + 24\), find the zeroes and use them to write the equation in factored form, then graph both the expanded and factored form to determine if they are the same equation. Review the use of \(y\) or \(f(x)\). Use the calculator to find \(f(4)\) and \(f(2)\).
  \[\text{Solution: } f(x) = (x - 4)(x + 3)(x - 2)\]
- Have students graph \(y = (x - 2)^2(x + 6)\) and find the zeroes. Discuss the difference between root and zero.
  \[\text{Solution: two zeroes } \{2, -6\}, \text{ three roots: } 2 \text{ is a double root and } 6 \text{ is a single root}\]
- **Graphs of Polynomials – Discovery Worksheet**
  Divide students into groups of three to graph complete this worksheet.

  Directions: Graph each of the equations graphing \(y_1, y_2\) and \(y_3\) on the same graph and adjust the window to display both intercepts and the maximum and minimum points. Find the zeroes and sketch the graph on your paper locating zeroes. Write the equations in factored form and check to see if both forms are equivalent. Answer the questions.

  1. \(y_1 = x^2 + 5x + 6, y_2 = 3x^2 + 15x + 18, y_3 = \frac{1}{2}x^2 + \frac{5}{2}x + \frac{6}{2}\). How many zeroes?
      How many roots? What is the effect a constant factor on the zeroes? What is the effect of the constant factor on the shape of the graph? Discuss end behavior. Graph \(y_4 = 3y_1\) (On calculator, find \(y_1\) under VARS/Y–VARS/1:Function/1:Y_1.) Which other equation is this equivalent to?
  2. \(y_1 = x^3 + 6x^2 + 8x, y_2 = -x^3 - 6x^2 - 8x\). How many zeroes? What is the effect of a common factor of \(-1\) have on the zeroes? on the end behavior?
  3. \(y_1 = x^2 - 6x + 9, y_2 = x^2 + 4x + 4\). How many zeroes are in each graph? How many roots? Discuss multiplicity. Discuss the looks of the graph.
  4. \(y_1 = x^3 - x^2 - 8x + 12\). How many zeroes? How many roots? Discuss multiplicity. Discuss end behavior.
  5. \(y_1 = x^4 - 3x^3 - 10x^2 + 24x\). How many zeroes? roots? Discuss end behavior. Looking at the number of roots in problems 1 through 4, how can you determine how many roots a polynomial has? Graph \(y_2 = -y_1\). What is the
effect on the zeroes and the end behavior? Looking at the end-behavior in problems 1 through 4, how can you predict end behavior?

6. \( y_1 = x^4 + 2x^3 - 11x^2 - 12x + 36 \). How many zeroes? How many roots? Discuss multiplicity. Discuss end behavior.

7. \( y_1 = x^5 - 6x^4 + 9x^3 \). How many zeroes? How many roots? Discuss multiplicity and end behavior. What is the difference in the looks of the graph for a double root and a triple root?

8. Find the zeroes and use the rules above to sketch a graph without a calculator:
   1) \( y = x^3 - 8x^2 + 16x \)
   2) \( y = -2x^2 - 14x - 10 \)
   3) \( y = (x - 4)(x + 3)(x + 1) \)
   4) \( y = -(x + 2)(x - 7)(x + 5) \)
   5) \( y = (2 - x)(3 - x)(5 + x) \)
   6) \( y = x^2 + 10 + 25 \)
   7) \( y = (x - 3)^3(x + 5) \)
   8) \( y = -(x - 3)^3(x + 5) \)
   9) \( y = (x - 3)^3(x + 5)^2 \)
  10) \( y = (x - 3)^4(x + 5)^2 \)

Solution:

1) Factored form: \( y_1 = (x + 3)(x + 2), y_2 = 3(x + 3)(x + 2), y_3 = \frac{1}{2}(x + 3)(x + 2) \), 2 zeroes at \( x = -2 \) and \( x = -3 \), 2 roots. The constant factor makes it steeper if it is \( > 1 \) and less steep if it is \( < 1 \). The graphs start up and end up. \( y_4 = y_2 \).

2) Factored form: \( y_1 = x(x + 4)(x + 2), y_2 = -x(x + 4)(x + 2) \), 3 zeroes \( \{0, -4, -2\} \). The negative leading coefficient does not change the zeroes but changes the direction of the graph – instead of starting down and ending up, it starts up and ends down.

3) Factored form: \( y_1 = (x - 3)^2, y_2 = (x + 2)^2 \), \( y_1 \) zero \( \{-3\} \), \( y_2 \) zero \( \{2\} \), both have two roots called double roots with a multiplicity of 2. The graph skims off the x–axis at the zero.

4) Factored form: \( y_1 = (x - 2)^2(x + 3) \), 2 zeroes \( \{-3, 2\} \), 3 roots – one double root at \( x = 2 \), starts down and ends up.

5) Factored form: \( y_1 = x(x - 4)(x + 3)(x - 2) \), four zeroes \( \{0, 4, -3, 2\} \), and four roots, \( y_1 \) starts up and ends up. The degree of the polynomial determines the numbers of roots. \( y_2 \) has the same zeroes but end behavior reverses. Even degree polynomials start and end in the same direction – positive up and negative down. Odd degree polynomials start and end in opposite directions – positive down then up and negative up then down.

6) Factored form: \( y_1 = (x - 2)^2(x + 3)^2 \), Two zeroes \( \{-3, 2\} \), four roots both zeroes are double roots. Starts up and ends up.

7) Factored form: \( y_1 = x^3(x - 3)^2 \), Two zeroes \( \{0, 3\} \), five roots with \( x = 0 \) a triple root and \( x = 3 \) a double root. Starts down and ends up. Even multiplicity skims off the x–axis and odd multiplicity passes through the x–axis flatter.
Activity 8: Modeling Real life Data with a Polynomial Function (GLEs: 1, 2, 4, 5, 6, 7, 8, 10, 16, 19, 22, 24, 25, 27, 28, 29)

The students will plot data in a scatter plot and determine what type of polynomial function best describes the data and create an equation based on the zeros.

Bellringer: Make a rough sketch of the graphs the following equations without a calculator:

1) \( f(x) = (x - 3)(x - 5)(x + 2) \)
2) \( f(x) = -3(x - 3)(x - 5)(x + 2) \)

Activity:
- Use the Bellringer to review the graphing procedure learned in Activity 7. Have students determine zeroes, end behavior and the effect of the leading constant, reinforcing that they can always plug in x-values to get a better shape.

- Using the following data, have students predict an equation that would model the data based on the zeroes. Have them plot the data in their calculators and enter the equation to see if it matches the data and then adjust the coefficients of the equation until they get a match. Solution: \( y = 2x(x - 3)(x + 2) \)

<table>
<thead>
<tr>
<th>(x)</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>-36</td>
<td>0</td>
<td>8</td>
<td>0</td>
<td>-12</td>
<td>-16</td>
<td>0</td>
<td>48</td>
</tr>
</tbody>
</table>

- Ask how students predicted a cubic with just the data available. Give the following data and ask if the data is linear and why. If so, find the equation of the line. Review slope and name the process of twice subtracting the \(y\)-values to get 0 (Algebra I, Unit 7, Activity 3), the Method of Finite Differences. Solution: \( f(x) = 3x - 5 \)

<table>
<thead>
<tr>
<th>(x)</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>-14</td>
<td>-11</td>
<td>-8</td>
<td>-5</td>
<td>-2</td>
<td>1</td>
<td>4</td>
<td>7</td>
</tr>
</tbody>
</table>

- Apply the method of finite differences to the first table several times and let the students develop the guidelines:
  1) If the 1st order differences are 0, then \( y = c \).
  2) If the 2nd order differences are 0, then \( y = ax + b \).
  3) If the 3rd order differences are 0, then \( y = ax^2 + bx + c \)
  4) If the 4th order differences are 0, then \( y = ax^3 + bx^2 + cx + d \)

- Discuss the limitation of using this method in evaluating real life data.

- Application Problem:
  Because of improved health care, people are living longer. The following data relates the number of Americans (in thousands) who are expected to be over 100 years old for the selected years. (Source: US Census Bureau)

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>number</td>
<td>50</td>
<td>56</td>
<td>65</td>
<td>75</td>
<td>94</td>
<td>110</td>
</tr>
</tbody>
</table>
Have students enter the data into the calculator, letting $x = 4$ correspond to 1994 and make a scatter plot. Then graph the following equations and determine which polynomial best models the number of Americans over 100 years old:

$y_1 = 6.057x + 20.4857$
$y_2 = 0.4018x^2 - 1.175x + 48.343$
$y_3 = -0.007x^3 + 0.5893x^2 - 2.722 + 52.1428.$

Use the equation chosen to predict the number of Americans who will be over 100 years old in the year 2008. (Discuss finding $f(18)$ on the calculator by several methods: tracing on the graph of $y_3$ to $x = 18$, on the home screen use $y_3(18)$, or use the table function.)

Solution: $y_3(18) = 153,571$ Americans

Activity 9: Solving Polynomial Inequalities (GLEs: 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 19, 24, 25, 27, 28, 29)

In this activity, students will solve polynomial inequalities using both a sign chart and graph.

Bellringer: Solve for $x$:

1. $-2x + 6 > 0$
2. $x(x - 4) > 0$
3. $x(x - 4) \leq 0$

Solutions: (1) $x < -3$
(2) $x < 0$ or $x > 4$
(3) $0 \leq x \leq 4$

Activity:

- Have students state the zero-product property. Most students will solve Bellringer problem 2 incorrectly, forgetting about the negative-times-negative solution. Ask students if $x = -5$ is a solution. Use the Bellringer to generate the discussion concerning the inequality properties:
  (1) $ab > 0$ if and only if $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$. (Review compound sentence use of and and or.)
  (2) $ab < 0$ if and only if $a < 0$ and $b > 0$ or $a > 0$ and $b < 0$.

- Solving Inequalities using a Sign Chart
Have students draw a number line and locate the zeroes for Bellringer problem 2. Reinforce that the zeroes are the values that divide the number line into intervals and satisfy the equation. Test values in each interval and write the solution in set notation and interval notation. Repeat with problem 3.

Solution: problem 2- interval notation: $(-\infty, 0) \cup (4, \infty)$,

problem 3- interval notation: $[0, 4]$
• Have students solve the following after discussing isolating 0 and not dividing by the variable (not only because the variable may be zero, but if it is negative it changes the inequality sign). Use guided practice for more polynomials. Write answers in set notation.

(3) \((x - 3)(x + 4)(x - 7) \geq 0\)
(4) \(x^2 - 9x < -14\)
(5) \(5x^3 \leq 15x^2\)

Solution: (3) \(-4 \leq x \leq 7\)
(4) \(2 < x < 7\)
(5) \(x \leq 3\)

• Critical Thinking Writing Activity

Have students graph the following equations with graphing calculators. Then have them develop a process to use the graph to find the solutions for the inequalities, write in interval notation, and write a paragraph explaining how and why you can determine the answer from the graph. (Solution same as problems above.)

(1) Graph \(y = -2x + 6\) to solve \(-2x + 6 > 0\)
(2) Graph \(y = x(x - 4)\) to solve \(x(x - 4) > 0\)
(3) Graph \(y = x(x - 4)\) to solve \(x(x - 4) \leq 0\)
(4) Graph \(y = (x-3)(x+4)(x - 7)\) to solve \((x - 3)(x + 4)(x - 7) \geq 0\)
(5) Graph \(y = x^2 - 9x + 14\) to solve \(x^2 - 9x < -14\)
(6) Graph \(y = 5x^3 - 15x^2\) to solve \(5x^3 \leq 15x^2\)

• Have the students share their thinking in the above activity with the class and then solve problems 7 and 8 by graphing and without using a graphing calculator:

(7) \(x(x - 4)(x + 6)(x - 2) > 0\)
(8) \(-2x(x - 3)^2(x + 4) \leq 0\)

Solution: (7) \((-\infty, -6) \cup (0, 2) \cup (4, \infty)\)
(8) \((-\infty, -4] \cup [0, \infty)\)

Sample Assessments

General Assessments

• The teacher will use Bellringers as ongoing informal assessments
• The teacher will collect the Little Black Book of Algebra II Properties and grade for completeness at the end of the unit
• The teacher will monitor student progress using small quizzes to check for understanding during the unit on such topics as the following:
  (1) expanding and factoring the difference in two perfect squares and cubes
  (2) factoring a trinomials
  (3) solving polynomial equations by factoring
  (4) factoring and graphing polynomials
  (5) solving polynomial inequalities
• The student will demonstrate proficiency on two comprehensive assessments:
  (1) factoring polynomial expressions
  (2) solving polynomial equations and inequalities and graphing
Activity-Specific Assessments

- **Activity 6**: The teacher will evaluate the *Translating Graphs of Lines – Discovery Worksheet* (see activity) using the following rubric:

  *Grading Rubric for Discovery Worksheets*
  
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  
  2 pts. - correct use of mathematical language
  
  2 pts. - correct use of mathematical symbols
  
  2 pts./graph - correct graphs and/or equations (if applicable)
  
  5 pts. - correct conclusion

- **Activity 7**: The teacher will evaluate the *Graphs of Polynomials – Discovery Worksheet* – (see activity) using the Grading Rubric for Discovery Worksheets in Activity 6.

- **Activity 9**: The teacher will evaluate the *Critical Thinking Writing Activity* (see activity) as indicated by the rubric below:

  *Grading Rubric for Critical Thinking Writing Activities*
  
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  
  2 pts. - correct use of mathematical language
  
  2 pts. - correct use of mathematical symbols
  
  3 pts./graph - correct graphs (if applicable)
  
  3 pts./solution - correct equations, showing work, correct answer
  
  3 pts./discussion - correct conclusion
Algebra II
Unit 3: Rational Equations and Inequalities

Time Frame: Approximately three weeks

Unit Description

The study of rational equations reinforces the students’ factoring and expanding skills. This unit develops the process for simplifying rational expressions, adding, multiplying and dividing rational expressions, and solving rational equations and inequalities.

Student Understandings

Students should understand how to symbolically manipulate rational expressions in order to solve rational equations. They should determine the domain restrictions that drive the solutions of rational functions. They should relate the domain restrictions to vertical asymptotes on a graph of the rational function but realize that the calculator does not give an easily readable graph of rational functions. Therefore, they must be able to solve rational inequalities by the sign chart method instead of the graph. Students also solve application problems involving rational functions.

Guiding Questions

1. Can students simplify rational expressions in order to solve rational equations?
2. Can students add, subtract, multiply, and divide rational expressions?
3. Can students simplify a complex rational expression?
4. Can students solve rational equations?
5. Can students identify the domain and vertical asymptotes of rational functions?
6. Can students solve rational inequalities?
7. Can students solve real world problems involving rational functions?

Unit 3 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
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<tr>
<td>Number and Number Relations</td>
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<tr>
<td>1.</td>
<td>Read, write, and perform basic operations on complex numbers (N-1-H) (N-5-H)</td>
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<td>Algebra</td>
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<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
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<td>5.</td>
<td>Factor simple quadratic expressions including general trinomials, perfect squares, difference of two squares, and polynomials with common factors (A-2-H)</td>
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<td>GLE #</td>
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<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
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<td>7.</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)</td>
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<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td>9.</td>
<td>Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-4-H)</td>
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<tr>
<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
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**Geometry**

| 16.   | Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H) |

**Patterns, Relations, and Functions**

| 24.   | Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H) |
| 25.   | Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H) |
| 27.   | Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H) |
| 28.   | Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H) |
| 29.   | Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H) |

**Sample Activities**

**Ongoing:** **Little Black Book of Algebra II Properties - Rational Equations and Inequalities**

The following is a list of properties to be included in the Little Black Book of Algebra II Properties. Add other items as appropriate.

3-1 **Rational Terminology** – define *rational number, rational expression, and rational function, least common denominator (LCD), complex rational expression.*

3-2 **Rational Expressions** – explain the process for simplifying, adding, subtracting, multiplying, and dividing rational expressions; define *reciprocal,* and explain how to find denominator restrictions.

3-3 **Complex Rational Expressions** – define and explain how to simplify.

3-4 **Vertical Asymptotes of Rational Functions** – explain how to find domain restrictions and what the domain restrictions look like on a graph, explain how to determine end-behavior of a rational function around a vertical asymptote.
3-5 **Solving Rational Equations** – explain the difference between a rational expression and a rational equation, list two ways to solve rational equations, define *extraneous roots*.

3-6 **Solving Rational Inequalities** - list the steps for solving an inequality by using the sign chart method.

**Activity 1: Simplifying Rational Expressions (GLEs: 1, 5, 7)**

In this activity, the students will review non-positive exponents and use their factoring skills from the previous unit to simplify rational expressions. It is important for future mathematics courses that students find denominator restrictions throughout this unit. They should never write \( \frac{x}{x} = 1 \) unless they also write *if* \( x \neq 0 \) because the two graphs of these functions are not equivalent.

**Bellringer:** Simplify:

\[ (1) \left( x^2y^5 \right)^4 \]
\[ (2) \left( \frac{x^5}{x^3} \right)^3 \]
\[ (3) \frac{x^7}{x^7} \]
\[ (4) \frac{x^3}{x^5} \]

*Solution:* (1) \( x^8y^{20} \), (2) \( x^7 \), \( x \neq 0 \), (3) 1, \( x \neq 0 \), (4) \( \frac{1}{x^2} \)

**Activity:**

- Use the Bellringer to review Laws of Exponents and let the students develop the meaning of zero and negative exponents. Connect negative exponents to what they have already learned about scientific notation in Algebra I and science. To reinforce the equivalencies, have students enter the following in their calculators on the home screen:

  1) \( 3^0 \)
  2) \( 2^{-3} \text{ and } \frac{1}{2^3} \)
  3) \( .001 \text{ and } \frac{1}{10^3} \text{ and } 10^{-3} \)
  4) \( .00037 \text{ and } 3.7 \times 10^{-4} \)

- Use guided practice with problems in which students simplify and write answers with only positive exponents such as

  \[ (1) \left( x^{-3} \right)^4 \]
  \[ (2) \left( x^{-2} \right)^3 \]
  \[ (3) \frac{x^{-3}}{x^2} \]

*Solution:* (1) \( x \), (2) \( \frac{1}{x^6} \), (3) \( \frac{1}{x} \)
• Have students define rational number to review the definition as the quotient of two integers \( \frac{p}{q} \) in which \( q \neq 0 \) and then define rational algebraic expression as the quotient of two polynomials \( P(x) \) and \( Q(x) \) in which \( Q(x) \neq 0 \). Discuss the restrictions on the denominator and have students find the denominator restrictions on the following rational expressions.

\[
\frac{4x^3}{7t} \cdot \frac{3x + 5}{y - 3}, \frac{2x + 5}{3x - 7}, \frac{3x + 2}{x^2 + 5x + 6}, \frac{4}{x^2 - 9}
\]

**Solutions:**
1) \( t \neq 0 \)
2) \( y \neq 3 \)
3) \( x \neq \frac{7}{3} \)
4) \( x \neq 2,3 \)
5) \( x \neq \pm3 \)

Have students simplify \( \frac{24}{40} \) and let one student explain the steps he/she used. Make sure there is a discussion of dividing out of a common factor. Then have students apply this concept to simplify the following expressions and develop the process to simplify rational expressions. Specify all denominator restrictions.

\[
\frac{-27x^2y^4}{9x^4y}, \frac{a-b}{b-a}, \frac{8-2x}{x^2-5x+6}, \text{ and } 8x(4x-28)^{-1}.
\]

• Remind students that denominator restrictions apply to the original problem, not the simplified problem. To stress this point, have the students simplify

\[ f(x) = \frac{x^3 - 2x^2 + 4x - 8}{x^2 - 2} \]

and graph both on the graphing calculator. Trace to \( x = 2 \) on both to find \( f(2) \) and \( g(2) \). There is a hole in one graph and not in the other; therefore, they are only equal for all values of \( x \) except \( x = 2 \). Verify this in a table: go to Table Set (TBL SET) and start = 0 and go up by increments (Tbl) = 0.2. Again you will see no value for \( x = 2 \).

• **Application:**
The side of a regular hexagon is \( 2a^2b^3 \) and the side of a regular triangle is \( 3a^2b^3 \). Find the ratio of the perimeter of the hexagon to the perimeter of the triangle.

**Solution:**
\[ \frac{4b^2}{3a} \]
Activity 2: Multiplying and Dividing Rational Expressions (GLEs: 1, 5)

In this activity the students will multiply and divide rational expressions and use their factoring skills to simplify the answer and express domain restrictions.

Bellringer: Simplify:

1) \( \frac{3}{4} \cdot \frac{10}{11} \)
2) \( \frac{7}{8} \cdot 4 \)
3) \( \frac{4x^2}{5y} \cdot \frac{y^3}{12x^5} \)
4) \( \frac{x+2}{x-3} \cdot \frac{4}{5} \)
5) \( \frac{2x+3}{x-5} \cdot (x-2) \)
6) \( \frac{x-2}{x+4} \cdot \frac{x+3}{x-5} \)

Solutions:
1) \( \frac{15}{22} \)
2) \( \frac{7}{2} \)
3) \( \frac{y^2}{15x^3} \)
4) \( \frac{4x+8}{5x-15} \)
5) \( \frac{2x^2-x-6}{x-5} \)
6) \( \frac{x^2+x-6}{x^2-x-20} \)

Activity:
- Use the Bellringer to review the process of multiplying numerical fractions and have students extend the process to multiplying rational expressions. Students should simplify and state denominator restrictions.
• Have students multiply and simplify \( \frac{x^2 - 4}{x + 3} \cdot \frac{2x + 6}{x^2 + 7x + 10} \) and let students that have different processes show their work on the board. Examining all the processes, have students choose the most efficient (factoring, canceling, and then multiplying). Make sure to include denominator restrictions.

• Have the students work the following \( \frac{3}{4} \div \frac{10}{11} = \) and \( \frac{7}{8} \div 4 \). Define reciprocal and have students rework the Bellringers with a division sign instead of multiplication.

• Application
Density is mass divided by volume. The density of solid brass is \( \frac{x + 5}{2} \text{ g/cm}^3 \). If a sample of an unknown metal in a laboratory experiment has a mass of \( \frac{x^2 + 2x - 15}{2x - 8} \text{ g} \) and a volume of \( \frac{x^2 + x - 12}{x^2 - 16} \text{ cm}^3 \), determine if the sample is solid brass.

Solution: yes

• Critical Thinking Writing Activity
(1) Describe what is similar about simplifying both expressions:
\[
\frac{42}{72} = \frac{7}{12} \quad \text{and} \quad \frac{x^2 - 2x}{12x + 12} \cdot \frac{7x^2 + 21x + 14}{x^3 - 4x} = \frac{7}{12}
\]
(2) Describe what the student did wrong in the following problem. Describe how you would find the correct reciprocal. Find the correct reciprocal and simplify it.

The reciprocal of \( \left( \frac{x - 7}{x} \right) \) is \( \left( \frac{5 - x}{7} \right) \).

Activity 3: Adding and Subtracting Rational Expressions (GLEs: 1, 5, 10, 24, 25)

In this activity, the students will find common denominators to add and subtract rational expressions.

Bellringer: Simplify and express answer as an improper fraction:

(1) \( \frac{2}{5} + \frac{7}{11} \quad \text{Solution:} \quad \frac{57}{55} \)
(2) \( \frac{2}{15} + \frac{7}{25} \)
(3) \( \frac{2}{5} + 6 \)

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Activity:

- Use the Bellringer to develop the rules for adding rational expressions. Have students apply this process to find the sums:
  
  \[
  (1) \quad \frac{x - 5}{6x^2} + \frac{2x + 6}{6x^2} \\
  (2) \quad \frac{x - 5}{6x^2 - 54} + \frac{2x + 6}{x - 3}.
  \]

  Solutions:
  
  \[
  (1) \quad \frac{3x + 1}{6x^2}, \quad (2) \quad \frac{12x^2 + 73x + 103}{6x^2 - 54}
  \]

- Have students subtract and simplify \(\frac{2}{15} - \frac{7}{25}\) and \(\frac{2}{5} - 6\). Have them work the following:
  
  \[
  (1) \quad \frac{x - 5}{6x^2} - \frac{2x + 6}{6x^2} \\
  (2) \quad \frac{x - 5}{6x^2 - 54} - \frac{2x + 6}{x - 3}
  \]

  Solutions:
  
  \[
  1) \quad -\frac{2x - 11}{6x^2} \\
  2) \quad -\frac{12x^2 - 71x - 113}{6x^2 - 54}
  \]

- 1st Critical Thinking Writing Activity

  The time it takes a boat to go downstream is represented by the function
  
  \(d(x) = \frac{2}{x + 1}\) hours, where \(x\) represents the number of miles. The time it takes a boat to go upstream is represented by the function \(u(x) = \frac{3}{x - 1}\) hours.

  a. How long in minutes does it take to go 2 miles upstream? 2 miles downstream? Explain why it would be different?
  
  b. Find a rational function \(f(x)\) for the total time in minutes. Then find the total time it takes to go a total of 2 miles upstream then back to the starting point.
  
  c. Find a rational function \(g(x)\) for how much more time it takes to go upstream than downstream. Then find how much more time in minutes it takes to go upstream than downstream if you have traveled 2 miles upstream and back to the starting point.

  Solutions:
  
  a. \(u(2) = 3\) hours = 180 minutes, \(d(2) = 40\) minutes, current helps going downstream
b. \( u(x) + d(x) = f(x) = \frac{5x + 1}{x^3 - 1} \), \( f(2) = 220 \) minutes

c. \( u(x) - d(x) = g(x) = \frac{x + 5}{x^2 - 1} \), \( g(2) = 140 \) minutes

• 2nd Critical Thinking Writing Activity:
  (1) Describe what common process is used to find both sums:
  \[
  \frac{5}{12} + \frac{4}{15} = \frac{41}{60} \quad \text{and} \quad \frac{5}{2x^3 y} + \frac{4}{3xy^3} = \frac{15y^2 + 8x}{6x^2 y^3}
  \]

  (2) Describe what the student did wrong in the following problem and how to solve it correctly.
  \[
  \frac{a}{c} - \frac{b - d}{c} = \frac{a - b - d}{c}
  \]

Activity 4: Complex Rational Expressions (GLEs: 1, 5)

The students will simplify complex rational expressions.

Bellringer: Multiply and simplify the following:

1. \( 6x^2 y^2 \left( \frac{x}{6y^2} + \frac{3y}{2x^2} \right) \)
2. \( (x + 2)(x - 5) \left( \frac{3}{x + 2} + \frac{7}{x - 5} \right) \)

Solutions:
1. \( y^3 + 9y^3 \)
2. \( 10x - \)

Activity:

• Use the Bellringer to review the distributive property.

• Define complex fraction and ask students how to simplify \( \frac{1}{65} \). Most will invert and multiply. Discuss an alternate process of multiplying by 18/18 or the LCD ratio equivalent to 1.

• Define complex rational expression and have students determine the best way to simplify \( \frac{1 + 4}{5 + \frac{3}{y}} \). Discuss the Activity 1 - Writing Activity 2 and why it would be wrong to work this problem this way: \( \left( \frac{1}{x + 4} \right) \left( \frac{1}{5} + \frac{y}{3} \right) \).
• Have students determine the process to simplify \( \frac{\frac{2}{x+3} + \frac{5x}{4}}{\frac{x^2-9}{x+3} + \frac{2}{x-3}} \)

\textit{Solution:} \( \frac{7x-6}{6x-6} \)

• \textbf{Critical Thinking Writing Activity}
Put students in groups of three to complete a worksheet of ten practice problems. Have each student take one of the problems and write a verbal explanation of the step-by-step process used to simplify the problem including all the properties used and why.

\textbf{Activity 5: Solving Rational Equations (GLEs: 1, 4, 5, 6)}

The students will solve rational equations.

\textbf{Bellringer:} Solve: \( \frac{x}{2} + \frac{3x}{4} = 5 \)

\textit{Solution:} \( x = 4 \)

\textbf{Activity:}
• Use the Bellringer to discuss alternate ways to solve this linear equation:
  (1) finding LCD and adding fractions
  (2) multiplying both side of the equation by the LCD to remove fractions then solve for \( x \). Always check the solution because the answer may be a solution to the transformed equation but not the original equation.

• Have students solve and check the following:
  (1) \( \frac{1}{4x} - \frac{3}{4} = \frac{7}{x} \)
  (2) \( \frac{x}{x-2} = \frac{1}{2} + \frac{2}{x-2} \)

\textit{Solution} (1) \( x = -9 \), (2) \textit{no solution}, 2 is an extraneous root

• Use the following solutions to develop the concept of zeroes of the function. Find the denominator restrictions and the solutions for the following:
  (1) \( \frac{x-2}{x+3} = 0 \)
  (2) \( \frac{x^2-x+12}{3x^2} = 0 \)

\textit{Solution:} (1) \( x = -4, x = 4, x \neq 0 \), (2) \( x = 2, x \neq -3 \)
• **Critical Thinking Writing Activity**
  Every camera lens has characteristic measurement called focal length, $F$. When the object is in focus its distance, $D$, from the lens and the distance, $L$, from the lens to the film satisfies the following equation. $\frac{1}{L} + \frac{1}{D} = \frac{1}{F}$. If the distance from the lens to an object is 60 cm and the distance from the lens to the film is 3 cm greater than the focal length, what is the focal length of the lens? Draw a picture of the subject, the film, and the lens and write the variables on the picture. Set up the equation and solve. Discuss the properties you used and if the answers you got are feasible and why. *Solution: 12 cm*

![Diagram of subject, film, and lens](image)

**Activity 6: Applications Involving of Rational Expressions (GLEs: 1, 5, 9, 10, 29)**

In this activity, students will solve rate problems that are expressed as rational equations.

**Bellringer:** In this class, 2 out of 5 of you are wearing blue. If 14 of you are wearing blue, how many are there in the class? Set up a rational equation and solve.

*SOLUTION:* $\frac{2}{5} = \frac{14}{x}$, $x=35$

**Activity:**

- Use the Bellringer to review the meaning of ratio and proportion. Ask the students what is the rate of blue wearers to any color wearers and define rate as a comparison of two quantities with different units and proportion as an equation setting two rates equal to each other (with the units expressed in the same order).

- Give the students the following example to set up a rational equation and solve. John’s car uses 18 gallons to travel 300 miles. He has 7 gallons of gas in the car and wants to know how much more gas will be needed to drive 650 miles. Assuming the car continues to use gas at the same rate, how many more gallons will be needed?

*SOLUTION:* $\frac{18}{300} = \frac{7 + x}{650}$, 32 gallons
• Give the students the following Algebra I type problem:

Jerry walks 5 miles per hour and travels for 3 hours. How far does he walk?

Use the problem to discuss distance, rate, and time and write a rational equation solved for time. \( t = \frac{d}{r} \). Then solve the following problem:

Sue and Bob are walking at the same speed. Bob jumps on a moving sidewalk that travels 3 feet per second and continues to walk at the same rate the 600 feet to the airplane door. He beats Sue by 180 seconds. What was their walking rate?

Solution: Sue’s time = \( \frac{600}{r} \), Bob’s time was \( \frac{600}{r+3} \). Since Bob’s time is 180 seconds less than Sue’s time the rational equation is \( \frac{600}{r} - 180 = \frac{600}{r+3} \), \( r = 2 \) feet per second

• Have the students list the 6-step process for solving application problems developed in Unit 1.
   (1) Define the variables and the given information
   (2) Determine what you are asked to find
   (3) Write an equation.
   (4) Solve the equation
   (5) Check
   (6) Answer the question in a sentence making sure to include units

• Give the students the following Algebra I type problem:

Mary plants flowers at a rate of 200 seeds per hour. How many seed has she planted in 2 hours?

Use the problem to discuss the formula, Amount of work \( (A) = \) rate \( (r) \) times time \( (t) \), rewriting the equation as the rational equation \( r = \frac{A}{t} \). If one whole job can be accomplished in \( t \) units of time, then the rate of work is \( \frac{1}{t} \). Use this concept to solve the following problem:

Harry and Melanie are working on Lake Ponchartrain cleanup detail. Harry can clean up the trash in his area in 6 hours and Melanie can do the same job in 4 hours. How long will it take them to clean that area if they worked together?

Solution: \( \frac{1}{6} + \frac{1}{4} = \frac{1}{t} \), 2.4 hours
• Put students in pairs to set up equations for additional problems that they will solve at home.

Activity 7: Vertical Asymptotes on Graphs of Rational Functions (GLEs: 1, 4, 5, 6, 7, 9, 10, 16, 25, 27, 28)

In this activity, students will use technology to look at the graphs of rational functions in order to locate vertical asymptotes and relate them to the domain restrictions. (Finding horizontal asymptotes and graphing rational functions is a skill left to Advanced Math because of its relationship to limits.)

Bellringer:
Find the solution and the denominator restrictions for the following rational equation.
\[
\frac{2x - 6}{x + 2} = 0
\]

Solution: \(x = 3, \ x \neq -2\)

Activity:
• Have the students graph \(f(x) = \frac{2x - 6}{x + 2}\) from the Bellringer on their graphing calculators and ask them to find the zero of the graph. Ask, What do you see at \(x = -2\)? Have students find \(f(-2)\) which has no \(y\) value. Change the calculators from connected mode to dot mode to show that there really is no graph at \(x = -2\). Define asymptote and demonstrate how to draw a dotted line at the vertical asymptote on a graph.

• End-Behavior Around Vertical Asymptotes – Discovery Worksheet

Directions: Graph the following equations on your graphing calculator and sketch the graphs. All the functions have a horizontal asymptote at \(y = 0\). Draw a dotted line at this place. Make sure to draw a dotted line where you expect the vertical asymptote to be. Write the equation of the vertical asymptote.

1. \(y = \frac{1}{x - 2}\)
2. \(y = \frac{1}{(x - 2)^2}\)
3. \(y = \frac{1}{(x - 2)^3}\)
4. \(y = \frac{1}{(x - 2)^4}\)
5. \(y = \frac{1}{(x - 2)(x + 3)}\)
6. \(y = \frac{-1}{(x - 2)^2(x + 3)}\) (Hint: Zoom box around \(x = -3\) to check graph)

Use the results from the graphs to answer the following questions:
7. How do you determine where the vertical asymptote is?
(8) What are the equations for the following vertical asymptotes?

(a) \( y = \frac{1}{(x - 4)(x + 3)(x - 2)^2} \)
(b) \( y = \frac{1}{2x^2 + 7x - 15} \)

(9) What effect does the degree of the factor in the denominator have on the graph?

(10) Predict the graphs of the following equations and then check on your calculator:

(a) \( y = \frac{40}{(x - 5)^2(x - 8)^3} \)
(b) \( y = \frac{1}{x + 2} \)
(c) \( y = \frac{-1}{x + 2} \)

Solution: (7) Set the denominator = 0 and solve for x. (8a) x = 4, x = -3, x = 2, (8b) x = 1.5, x = -4. (9) If the factor is even, the y approaches the same infinity around the asymptote. If the factor is odd, the y approaches opposite infinities around the asymptote.

Activity 8: Rational Equation Lab - “Light at a Distance” (GLEs: 1, 4, 6, 7, 8, 10, 25, 29)

It is important that students get to experience the use of rational functions in applications. In the lab listed below, the students use a light sensor along with a CBL unit to record light intensity as the sensor moves away from the light bulb.

Lab: Adapted from “Light at a Distance”, Real World Math with the CBL System by Chris Bruengsen et al. Texas Instruments Incorporated (1994).

Instructions: Mark off 10-centimeter intervals up to 2 meters from a light socket. The light sensor should be pointed directly at the illuminated bulb with the end of the sensor held at a certain distance from the bulb. The room should be darkened with the exception of the light bulb. Start the BULB program and follow directions on the program. The data plot should show intensity values that decrease as the distance increases.

Activity: Model the data collected by using a rational equation in the form \( f(x) = \frac{A}{x^n} \).

In order to find A, trace to the point where \( x = 1 \), find the y-value. Substitute the x and y value in the general equation and solve for A. Trace to another point and substitute x and y and A values. To solve for B on your calculator, set the expression = 0 and use the SOLVE feature on your calculator by putting the command SOLVE (expression, B, guess) on your home screen. (Enter any value such as 2 or 3 for the guess.) Type your new equation in \( y_1 \) on your calculator and see how well it follows the data. Compare this to the regression equation that your calculator will generate for this data. STAT/Calc/Pwreg/L1, L2, \( y_1 \). Which graph best follows the data?

Questions:
(1) How would a brighter or dimmer light bulb affect the values of A and B, if at all?
(2) Extrapolate on your equation to find the intensity of the light bulb at 3 meters. How would this be written in function notation?
(3) Find \( f(1.5) \). What does this mean in real world terms?
(4) What does the graph of this function look like on the x interval of [–10,10]?
(5) Is there real-world meaning to the portion of the graph where x < 0?
(6) What is happening on the graph at x = 0? What is the real-world meaning to this?
(7) What is happening on the graph as x approaches infinity? Interpret its real-world meaning. Is this realistic?

Alternate Projects if CBL equipment is unavailable:
• Whelk-Come to Mathematics: Using Rational Functions to Investigate the Behavior of Northwestern Crows, http://illuminations.nctm.org/index_o.aspx?id=143 – Students make conjectures, conduct an experiment, analyze the data, and work to a conclusion using rational functions to investigate the behavior of Northwestern Crows.
• Alcohol and Your Body by Rosalie Dance and Hames Sandifer (1998), http://www.georgetown.edu/projects/handsonmath/downloads/alcohol.htm - Students use rational functions to model elimination of alcohol from the body and learn to interpret horizontal and vertical asymptotes in context.

Activity 9: Rational Inequalities (GLEs: 1, 4, 5, 6, 7, 8, 9, 10, 16, 24, 25, 27, 28, 29)

In this activity, the students will solve rational inequalities using the sign chart.

Bellringer: Solve for x using a sign chart: $(x – 3)(x + 4)(x – 5) \geq 0$. Explain why $x = –2$ is a solution to the inequality even though $–2 < 0$ when the problem says “$> 0$”.
(Solution: $[–4,3] \cup [5,\infty)$)

Activity:
• Use the Bellringer to review the concept that the zeroes create endpoints to the intervals of possible solutions to polynomial inequalities. Have students locate the zeroes on a number line and check numbers in each interval. Determine that the students’ explanations include a review of the properties of inequalities.
  1. $ab > 0$ if and only if $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$. (Review compound sentence use of and or.)
  2. $ab < 0$ if and only if $a < 0$ and $b > 0$ or $a > 0$ and $b < 0$.

• Have the students solve the following inequality: $\frac{2x + 10}{(x – 2)(x + 3)} \geq 0$. They will usually answer $x > –5$. Ask if $x = 1$ is a solution. Have students conclude that the sign chart should be used with intervals created by the zeroes and the denominator restrictions, so the solution is $[–5, –3] \cup [–5,–3) \cup (2, \infty)$. Discuss inclusion or noninclusion of endpoints.

• Have students solve $\frac{-1}{x – 3} \geq 1$. Discuss why a student cannot multiply both sides of the equation by $x–3$ to solve. It is because the sign would change if the denominator were negative. Have students understand that it must be changed to an equation which
can be solved by multiplying through by the LCD. Mark the number line intervals with the zeroes and the denominator restrictions.

- Have students develop the steps for solving rational inequalities:
  1. Change the inequality to an equation and solve by isolating zero.
  2. Set the denominator equal to 0 and solve the equation.
  3. Use the solution to step 1 and 2 to divide the number line into regions.
  4. Find the intervals that satisfy the inequality.
  5. Consider the endpoints and exclude any values that make the denominator zero.

- Have students graph $y = \frac{2x + 10}{(x - 2)(x + 3)}$ and discuss that since graphs of rational functions are not easily graphed by hand, finding the solution intervals is easier on the sign chart. Use guided practice to practice this skill.

- **Application**
  The production of heating oil produced by an oil refinery depends on the amount of gasoline produced. The amount of heating oil produced (in gallons per day) is modeled by the rational function $h(g) = \frac{125,000 - 25g}{125 + 2g}$ where $g$ is the amount of gasoline produced (in hundreds of gallons per day). If customers need more than 300 gallons of heating oil per day, how many gallons of gasoline will be produced?

  *Solution: less than 199.4 gallons of gasoline*

**Sample Assessments**

**General Assessments**

- The teacher will use Bellringers as ongoing informal assessments
- The teacher will collect the Little Black Book of Algebra II Properties and grade for completeness at the end of the unit.
- The teacher will monitor student progress using small quizzes to check for understanding during the unit on such topics as the following:
  1. multiplying and dividing rational expressions
  2. adding and subtracting rational expressions
  3. solving rational equations
  4. finding domain and vertical asymptotes
- The student will demonstrate proficiency on two comprehensive assessments:
  1. adding, subtracting, multiplying, dividing, and simplifying rational expressions, specifying denominator restrictions
  2. solving rational equations and inequalities, finding vertical asymptotes and matching graphs of rational functions
Activity-Specific Assessments

- **Activity 2**: The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the following rubric:

  *Grading Rubric for Critical Thinking Writing Activities:*
  - 2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  - 2 pts. - correct use of mathematical language
  - 2 pts. - correct use of mathematical symbols
  - 3 pts./graph - correct graphs (if applicable)
  - 3 pts./solution - correct equations, showing work, correct answer
  - 3 pts./discussion - correct conclusion

- **Activity 3**: The teacher will evaluate the 1st Critical Thinking Writing Activity (see activity) using rubric provided in the assessment for Activity 2. The teacher will evaluate the 2nd Critical Thinking Writing Activity (see activity) using the same rubric.

- **Activity 4**: The teacher will evaluate the Critical Thinking Writing Activity (see activity) using rubric provided in the assessment for Activity 2.

- **Activity 5**: The teacher will evaluate the Critical Thinking Writing Activity (see activity) using rubric provided in the assessment for Activity 2.

- **Activity 7**: The teacher will evaluate the End Behavior Around Vertical Asymptotes – Discovery Worksheet (see activity) using the following rubric:

  *Grading Rubric for Discovery Worksheets*
  - 2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  - 2 pts. - correct use of mathematical language
  - 2 pts. - correct use of mathematical symbols
  - 2 pts./graph - correct graphs and equations (if applicable)
  - 5 pts/discussion - correct conclusions

- **Activity 8**: The teacher will evaluate the Lab Report for “Light at a Distance” (see activity) using the rubric below:

  *Grading Rubric for Labs –*
  - 10 pts/ question - correct graphs and equations showing all the work
  - 2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  - 2 pts. - correct use of mathematical language
  - 2 pts. - correct use of mathematical symbols
Algebra II
Unit 4: Radicals and the Complex Number System

Time Frame: Approximately three weeks

Unit Description

This unit expands on the 9th and 10th grade GLEs regarding simplification of radicals with numerical radicands to include adding, subtracting, multiplying, dividing, and simplifying radical expressions with variables in the radicand and solving equations containing radicals. The unit also includes the development of the complex number system in order to solve equations with imaginary roots.

Student Understandings

Students will simplify radicals containing variables and solve equations containing radicals. Students will understand the makeup of the complex number system, and identify and classify each subgroup of numbers. Students will connect the factoring skills developed in Unit 2 to finding complex roots. They will understand the role of imaginary and irrational numbers in mathematics and when to use decimal approximations versus exact solutions. Upon investigation of the graphs of equations containing radicals and polynomials with imaginary roots, students should continue to develop the concepts of zeroes, domain, and range and use these to explain real and imaginary solutions and extraneous roots.

Guiding Questions

1. Can students simplify complex radicals having various indices and variables in the radicand?
2. Can students solve equations containing radicals and model real world applications as a radical equation and solve?
3. Can students explain extraneous roots with and without technology?
4. Can students classify numbers in the complex numbers system as rational, irrational, or imaginary?
5. Can students simplify expressions containing complex numbers?
6. Can students solve equations containing imaginary solutions?

Unit 4 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
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<tbody>
<tr>
<td>Number and Number Relations</td>
<td></td>
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<tr>
<td>1.</td>
<td>Read, write, and perform basic operations on complex numbers (N-1-H) (N-5-H)</td>
</tr>
<tr>
<td>GLE #</td>
<td>GLE Text and Benchmarks</td>
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<tr>
<td>2.</td>
<td>Evaluate and perform basic operations on expressions containing rational exponents (N-2-H)</td>
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<tr>
<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>5.</td>
<td>Factor simple quadratic expressions including general trinomials, perfect squares, difference of two squares, and polynomials with common factors (A-2-H)</td>
</tr>
<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
</tr>
<tr>
<td>7.</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)</td>
</tr>
<tr>
<td>9.</td>
<td>Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-4-H)</td>
</tr>
<tr>
<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
</tr>
<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td>24.</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
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<tr>
<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>28.</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
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Sample Activities

Ongoing: Little Black Book of Algebra II Properties - Radicals and the Complex Number System

The following is a list of properties to be included in the Little Black Book of Algebra II Properties. Add other items as appropriate.

4.1 Radical Terminology - define radical sign, radicand, index, like radicals, root, \( n^{th} \) root, principal root, conjugate.

4.2 Rules for Simplifying \( \sqrt[n]{b^n} \) - identify and give examples of the rules for even and odd values of \( n \).

4.3 Product and Quotient Rules for Radicals – identify and give examples of the rules.
4.4 **Rationalizing the Denominator** – explain: what does it mean, why do it, the process for rationalizing a denominator of radicals with varying indices and a denominator that contains the sum of two radicals.

4.5 **Radicals in Simplest Form** - list what to check for to make sure radicals are in simplest form.

4.6 **Addition and Subtraction Rules for Radicals** – identify and give examples.

4.7 **The Golden Ratio** – define the ratio, explain how found, identify the symbol, provide examples in the real world.

4.8 **Graphing Simple Radical Functions** – show the effect of constant both inside and outside of a radical on the domain and range.

4.9 **Steps to Solve Radical Equations** – identify and give examples.

4.10 **Complex Numbers** – define: \(i\), \(a + bi\) form, \(i, i^2, i^3, i^4\); explain how to find the value of \(i^{4n}, i^{4n+1}, i^{4n+2}, i^{4n+3}\); explain how to conjugate and find the absolute value of \(a + bi\).

4.11 **Properties of Complex Number System** – provide examples of the equality property, the commutative property under addition/multiplication, the associative property under addition/multiplication, and the closure property under addition/multiplication.

4.12 **Operations on Complex Numbers in a + bi form** – provide examples of addition, additive identity, additive inverse, subtraction, multiplication, multiplicative identity, squaring, division, absolute value, reciprocal, raising to a power, and factoring the sum of two perfect squares.

4.13 **Root vs. Zero** – explain the difference between a root and a zero and how to determine the number of roots of a polynomial.

**Activity 1: Roots and Radicals (GLEs 1, 2, 5, 7, 9, 10, 24, 25)**

The students will review the concepts of simplifying \(n^{th}\) roots and solving equations of the form \(x^n = k\) in order to develop the properties of radicals and simplify more complex radicals.

Emphasis in this lesson is on the new concept that \(\sqrt{x^2} = |x|\).

**Bellringer:** Graph on your graphing calculator and find the points of intersection:

1. \(y_1 = x^2\) and \(y_2 = 16\)
2. \(y_1 = x^2\) and \(y_2 = -16\)
3. \(y_1 = x^2\) and \(y_2 = 0\)

*Solution:* (1) \((\pm4,16)\), (2) empty set, (3) \((0, 0)\)

**Activity:**

- Use the Bellringer to generate a discussion about the number of answers for \(x^2 = 16, x^2 = -16, x^2 = 0\). Review the definition of *root* as the solution to an equation in one variable. Ask how this definition relates to the use of the word square root.

- Have the students define the terms *index, radical, radicand*. Have the students enter \(y_1 = \sqrt{x}\) and trace to \(x = 16\) in their calculators. There is one answer, 4, as opposed to the solution of #1 in the Bellringer, which has two answers, \(\pm4\). Discuss the definition of principle square root as the positive square root.
• Ask the students to solve the following:
  (1) $\sqrt[3]{6^2}$  (2) $\sqrt[3]{(-6)^2}$  (3) $\sqrt[3]{2^3}$  (4) $\sqrt[3]{(-2)^3}$

• Discuss and ask students to solve (5) $\sqrt{x^2}$ and (6) $\frac{1}{3}\sqrt[3]{x^3}$ and also enter $y = \sqrt{x^2}$ and $y = \frac{1}{3}\sqrt[3]{x^3}$ in their graphing calculator and identify the graphs as $y = |x|$ and $y = x$.

Review the piecewise function for $|x|$ and how it relates to $\sqrt{x^2}$.

• A very important concept for future mathematical study is the progression of the following solution. Explain and lead a class discussion about its importance.
  > $x^2 = 9$
  > $\sqrt{x^2} = \sqrt{9}$
  > $|x| = 3$
  > $x = \pm 3$

• Have the students solve some radical problems with different indices and develop the rules for $n^{th}$ root of $b^n$ where $n$ is even and $n$ is odd.
  
  1) If $n$ is even, then $\sqrt[n]{b^n} = |b| = \begin{cases} b & \text{if } b \geq 0 \\ -b & \text{if } b < 0 \end{cases}$
  
  2) If $n$ is odd, then $\sqrt[n]{b^n} = b$

• Have students determine how the $n^{th}$ root rule can be applied to expressions with multiple radicands such as $\sqrt[3]{8}x^3 = \sqrt[3]{(9x^3)^3} = 9x^2$ discussing why absolute value is not needed in this situation. Work and discuss $\sqrt[3]{64x^{15}} = \sqrt[3]{(4x^5)^3} = 4x^5$.

• Review the rules for solving absolute value equations and have students apply $n^{th}$ root rule to solve $\sqrt{(x + 3)^2} = 12$

  Solution: $\sqrt{(x + 3)^2} = 12 \Rightarrow |x + 3| = 9 \Rightarrow x = 6 \text{ or } -15$

• Application Problem

Meteorologists have determined that the duration of a storm is dependent on the diameter of the storm. The function $f(d) = 0.07(\sqrt{d})^3$ defines the relationship where $d$ is the diameter of the storm in miles and $f(d)$ is the duration in hours. How long will a storm last if the diameter of the storm is 9 miles? Write your answer in function notation with the answer in decimals and write your answer in a sentence in hours and minutes.

Solution: $f(9) = 1.89$, The storm will last approximately 1 hour and 53 minutes.
Activity 2: Multiplying and Dividing Radicals (GLEs 1, 2, 24, 25)

In this activity, the students will review the product and quotient rules for radicals addressed in Algebra I and use them to multiply, divide, and simplify radicals with variables in the radicand.

Bellringer: Simplify

(1) \(\sqrt{50}\)  
(2) \(\sqrt[3]{40}\)  
(3) \(\sqrt[3]{8}\)  
(4) \(\sqrt[3]{9}\)

Solution: (1) \(5\sqrt{2}\), (2) \(-2\sqrt[3]{5}\), (3) \(\frac{2\sqrt[3]{2}}{3}\), (4) \(\sqrt{3}\)

Activity:

• Have students put both the Bellringer problems and their answers in their calculators on the home screen to check for equivalency. This can be done by getting decimal representations or using the TEST feature of the calculator: Enter \(\sqrt[3]{50} = 5\sqrt[3]{2}\) (The = sign is usually found under TEST which is above the MATH button.) If the calculator returns a 1, then the statement is true; if it returns a 0, then the statement is false.

• Have students write the rule symbolically and verbally for multiplying and dividing radicals for radicals with the same indices by reviewing Bellringer problems 1, 2, and 3.

\[\sqrt[3]{a} \text{ and } \sqrt[3]{b}\] are real numbers and \(n\) is a natural number, then

1. \(\sqrt[3]{ab} = \sqrt[3]{a} \sqrt[3]{b}\). The radical of a product of equals the product of two radicals.

2. \(\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}\). The radical of a quotient is the quotient of two radicals, \(b \neq 0\).

• Discuss why the product rule does not apply in the following situation:

\[\sqrt[3]{(-4)(-9)} \neq \sqrt[3]{-4} \sqrt[3]{-9}\]

• Critical Thinking Writing Activity

Put students in pairs to check the answer for Bellringer problem 4 and to complete the writing activity below. Then have the pairs of students share their answers with the class to come up with a consensus.

What is the difference in rational and irrational numbers? Give definition and several examples of each in your discussion. Can the product of two irrational numbers be a rational number? What does “rationalizing the denominator” mean? Why do we rationalize the denominator? Explain how to rationalize the following denominators:

(1) \(\frac{1}{\sqrt{5}}\)  
(2) \(\frac{1}{\sqrt[3]{5}}\)  
(3) \(\frac{1}{\sqrt{8}}\)
• Have students list what should be checked to make sure a radical is in simplest form:
  (1) The radicand contains no exponent greater than or equal to the index,
  (2) the radicand contains no fractions
  (3) the denominator contains no radicals

• Put students back in their pairs to simplify the following expressions applying their
  rules to radicals with variables in the radicand.

  (1) \( \sqrt{72x^3y^4} \)  \hspace{1cm} (5) \( \frac{\sqrt{162x^6}}{\sqrt{10x^7}} \)
  (2) \( \frac{3}{\sqrt[3]{80s^4t}} \)
  (3) \( \sqrt{2xy} \cdot \sqrt{6x^3y} \)
  (6) \( \frac{3}{\sqrt{18s^3}} \)
  (4) \( 5\sqrt[4]{4x^2y^5} \cdot 7\sqrt{2x^2y} \)

• Application Problem:
The time in seconds, \( t(L) \), for one complete swing of a pendulum is dependent upon
the length of the pendulum in feet, \( L \), and gravity which is 32 ft/sec\(^2\) on earth. It is
modeled by the function \( t(L) = 2\pi \sqrt{\frac{L}{32}} \). Find the time for one complete swing of a
4-foot pendulum. Express the exact simplified answer in function notation and
express the answer in a sentence rounding to the nearest tenth of a second.

Solution:
(1) \( 6y^2|x|\sqrt{2x} \)
(2) \( 2st^2\sqrt[4]{10s} \)
(3) \( 2x^2|y|\sqrt{y} \)
(4) \( 70xy^2\sqrt[3]{x} \)
(5) \( \frac{9\sqrt{5x}}{5x} \)
(6) \( \frac{3\sqrt[3]{s^2}}{3s} \)
(7) \( t(L) = \frac{\pi \sqrt{2}}{2} \). One
complete swing of a
4-foot pendulum
takes approximately
2.2 seconds
Activity 3: Adding and Subtracting Radicals (GLEs 1, 2, 5)

In this activity, the students will review the sum and difference rules for radicals addressed in Algebra I and use them to add, subtract, and simplify radicals with variables in the radicand.

Bellringer: Simplify
1. \(6x^2 + 4y - x + 5x^2 - 7y + 9x\)
2. \(6\sqrt{2} + 4\sqrt{3} - \sqrt{2} + 5\sqrt{2} - 7\sqrt{3} + 9\sqrt{2}\)
3. \((3x + 5)(7x - 9)\)
4. \((3\sqrt{2} + 5)(7\sqrt{2} - 9)\)

Activity:
- Use the Bellringer to compare addition and multiplication of polynomials to addition and multiplication of radicals and how the distributive property is involved.
- Have students simplify the following to review to review a 9th Grade GLE: \(6\sqrt{18} + 4\sqrt{8} - 3\sqrt{72}\). Have students define like radicals as expressions that have the same index and same radicand and develop the rules for adding and subtracting radicals.
- Put students in pairs to use the rules they developed to simplify the following radicals:
  1. \(4\sqrt{18x} - \sqrt{72x} + \sqrt{50x}\)
  2. \(\sqrt{64xy^2} + \sqrt{27x^4y^5}\)
  3. \((2\sqrt{a} - 3\sqrt{b})(4\sqrt{a} + 7\sqrt{b})\)
  4. \((x + \sqrt{5})^2\)
  5. \((x + \sqrt{3})(x - \sqrt{3})\)

Solution
1. \(11\sqrt{2}x\), 2. \((4 + 3xy)\sqrt{xy^2}\), 3. \(8a + 2\sqrt{ab} - 21b\), 4. \(x^2 + 2x\sqrt{5} + 5\), 5. \(x^2 - 3\)

- Use problem 5 above to define conjugate and have students determine how to rationalize the denominator of \(\frac{1}{\sqrt{2} + \sqrt{5}}\).

- Critical Thinking Writing Activity:
  1. The product rule says that the radical of a product equals the product of the radicals. Discuss whether there is a sum rule that says that the radical of the sum equals the sum of the radicals. Give an example and discuss whether or not it is true and why.
2. Discuss whether the following is true: \( \sqrt{a^2 + b^2} = a + b \).

3. The Scarecrow in the 1939 movie *The Wizard of Oz* asked the Wizard for a brain. When the Wizard presented him with a diploma granting him a Th. D. (Doctor of Thinkology), the Scarecrow recited the following: “The sum of the square roots of the sides of an isosceles triangle is equal to the square root of the remaining side…” Did the Scarecrow recite the Pythagorean Theorem correctly? Explain.

**Activity 4: Graphing the Radical Function (GLEs 4, 6, 7, 16, 28)**

In this activity, the students will use technology to graph simple functions that involve radical expressions in preparation for solving equations involving radical expressions analytically. They will determine domain, range, and \( x \) and \( y \) intercepts.

**Bellringer:**

1. Graph \( f(x) = \sqrt{x} \) and \( g(x) = \sqrt[3]{x} \) on your graphing calculator and state the domain, range, and \( x \) and \( y \)-intercepts.

2. From the graph screen find \( f(3) \) and \( g(3) \) and round three decimal places. Are these answers rational or irrational numbers?

3. Graph \( y = 4 \) and use the intersection feature of your calculator to solve \( 4 = \sqrt{x} \) and \( 4 = \sqrt[3]{x} \).

**Activity:**

- Use the Bellringer to review graphing calculator skills. Divide the class into groups of three to complete the following worksheet.

- **Translating Radical Functions – Discovery Worksheet**
  
  I. Sketch the following graphs. State the domain and range and \( x \)- and \( y \)-intercepts.

     1. \( y = \sqrt{x} + 3 \)
     2. \( y = \sqrt{x} - 3 \)
     3. \( y = \sqrt[3]{x} + 2 \)
     4. \( y = \sqrt[3]{x} - 2 \)
     5. \( y = \sqrt{x} - 4 \)
     6. \( y = \sqrt{x} + 4 \)
     7. \( y = \sqrt[3]{x} + 5 \)
     8. \( y = \sqrt[3]{x} - 5 \)
     9. \( y = \sqrt{x - 3} + 5 \)

  II. Answer the following questions:

     10. What is the difference in the graph when a constant is added outside of the radical or inside of the radical?
(11) Compare the domain of all the functions containing $\sqrt{x}$ to the functions containing $\sqrt[3]{x}$. What is the difference in the domain and range of $f(x) = \sqrt{x}$? of $g(x) = \sqrt[3]{x}$?

(12) Why is the domain of one of the groups restricted and the other is unrestricted?

*Solutions:*

(1) $D: x \geq 0$, $R: y \geq 3$, $x$–intercept: none, $y$–intercept: $(0,3)$

(2) $D: x \geq 0$, $R: y \geq 0$, $x$–intercept: $(9,0)$, $y$–intercept: $(0,–3)$

(3) $D: all$ reals, $R: all$ reals, $x$–intercept: $(-8, 0)$, $y$–intercept: $(0, 2)$

(4) $D: all$ reals, $R: all$ reals, $x$–intercept: $(8, 0)$, $y$–intercept: $(0, –2)$

(5) $D: x \geq 4$, $R: y \geq 0$, $x$–intercept: $(4, 0)$, $y$–intercept: none,

(6) $D: x \geq 4$, $R: y \geq 0$, $x$–intercept: $(-4, 0)$, $y$–intercept: $(0, 2)$

(7) $D: all$ reals, $R: all$ reals, $x$–intercept: $(-5, 0)$, $y$–intercept: $\left(0, \sqrt[3]{5}\right)$

(8) $D: all$ reals, $R: all$ reals, $x$–intercept: $(5, 0)$, $y$–intercept: $\left(0, \sqrt[3]{-5}\right)$

(9) $D: x \geq 3$, $R: y \geq 5$, $x$–intercept: none, $y$–intercept: none

(10) Outside the radical changes the vertical shift, $+$ up and $-$ down. A constant inside the radical, changes the horizontal shift, $+$ left and $-$ right.

(11) Even index radicals have a restricted domain $x \geq 0$ and therefore a resulting restricted range $y \geq 0$. The domain and range of odd index radicals are both all reals.

(12) You cannot take an even radical of a negative number.

**Activity 5: Solving Equations with Radical Expressions (GLEs 1, 2, 4, 6, 7, 10, 16, 24, 28)**

In this activity, students will solve equations that involve radical expressions analytically as well as using technology and apply them to real-world applications.

**Bellringer:** Use your graphing calculator to graph the two functions and find the points of intersection in order to solve the following:

1. Graph $y_1 = \sqrt{3x-2}$ and $y_2 = 4$ to solve $\sqrt{3x-2} = 4$.

2. Graph $y_1 = \sqrt{3x-2}$ and $y_2 = -4$ to solve $\sqrt{3x-2} = -4$.

3. What is the difference?

   *Solution:*

   $1)$ $x=6$, $2)$ no solution, $3)$ A radical is never negative therefore there is no solution.

**Activity:**

- Ask the students to solve the following mentally and have them discuss their thought processes:

  $1)$ $\sqrt{x} - 4 = 0$  
  $2)$ $\sqrt{x} = 2$  
  $3)$ $\sqrt{x} = -5$
• Define and discuss *extraneous roots*. Use the discussion to generate steps to solve equations containing variables under radicals:
  1. Isolate the radical
  2. Raise both sides of the equation to a power that is the same as the index of the radical
  3. Solve
  4. Check

• Have students solve the following analytically:
  4. $\sqrt{3x - 2} = 4$
  5. $\sqrt{3x - 2} = -4$
  6. How are the problems above related to the graphs in the Bellringer?

• Have students solve the following analytically:
  7. $\sqrt{x - 3} + 5 = x$. (Review the process of solving polynomials by factoring and using the zero property.)
  8. Graph both sides of the equation (i.e. $y_1 = \sqrt{x - 3} + 5$ and $y_2 = x$) and explain why 7 is a solution and not 4.

• Solve and check analytically and graphically:
  9. $\sqrt{3x + 2} - \sqrt{2x + 7} = 0$
  10. $\sqrt{x - 5} - \sqrt{x} = 2$

  *Solution:* (9) 5, (10) no solution

• Application Problem
  The length of the diagonal of a box is given by $d = \sqrt{L^2 + W^2 + H^2}$ . What is the length, $L$, of the box if the height, $H$, is 4 feet, the width, $W$, is 5 feet and the diagonal, $d$, is 9 feet? Express your answer in a sentence in feet and inches rounding to the nearest inch.

  *Solution:* approximately 6 feet, 4 inches.

• Critical Thinking Writing Activity
  1. In solving radical equations, we have been squaring both sides of the equation and have not been concerned with the absolute value we used in previous lessons.
  Graph $y = \sqrt{x^2}$ and $y = \left(\sqrt{x}\right)^2$ on your graphing calculator. Sketch the graphs and explain the differences and why our process today has been accurate.
(2) Consider the radical \( \sqrt[m]{b^n} \). Determine whether the following are true or false.

(a) \( \sqrt{9^2} = \left( \sqrt{9} \right)^3 \)

(b) \( \sqrt[3]{8^2} = \left( \sqrt[3]{8} \right)^2 \)

(c) \( \sqrt{(-9)^4} = \left( \sqrt{-9} \right)^4 \)

(d) \( \sqrt[3]{(-27)^2} = \left( \sqrt[3]{-27} \right)^2 \)

(e) Explain when we apply the property \( \sqrt[m]{b^n} = \left( \sqrt[m]{b} \right)^n \)

**Activity 6: Imaginary Numbers (GLEs 1, 2, 4, 5, 6, 7, 9, 10, 16, 28)**

In this activity, students will develop the concept of imaginary numbers and determine their place in the complex number system. They will simplify square root radicals whose radicands are negative and rationalize the denominator of fractions with imaginary numbers in the denominator.

**Bellringer:**

I. Graph the following and find the zeroes:

1. \( y = x^2 - 4 \)
2. \( y = x^2 - 8 \)
3. \( y = x^2 \)
4. \( y = x^2 - 1 \)
5. \( y = x^2 + 1 \)

II. Solve the following analytically:

1. \( x^2 - 4 = 0 \)
2. \( x^2 - 8 = 0 \)
3. \( x^2 - 1 = 0 \)
4. \( x^2 = 0 \)
5. \( x^2 + 1 = 0 \)

**Activity:**

- Use the Bellringer to review the definition of zeroes, the number of roots of a polynomial, and a double root. Determine that \( \sqrt{-1} \) is the number we need to solve the following equation: \( x^2 + 1 = 0 \). Define that number as the number \( i \) in the set of Imaginary numbers which in union with the set of Real numbers make the set of Complex numbers. If \( \sqrt{-1} = i \), then \( i^2 = -1 \) and for all positive real numbers \( b \), \( \sqrt{-b} = i \sqrt{b} \).

- Have students simplify \( \sqrt{-7}, \sqrt{-16}, \sqrt{-24} \).
• Put students in pairs to determine the values of \( i^2, i^3, i^4, i^5, i^6, i^7, i^8, i^9 \) and have them write a rule that will help determine the answer to \( i^{27}, i^{37}, i^{42} \), and \( i^{20} \).

• Review the term \textit{rationalize the denominator} and discuss how it applies to problem in the form \( \frac{4}{\sqrt{-3}} \) and how to use the rules if \( i \) to rationalize this denominator. Return to pairs to rationalize the denominator of the following: \( \frac{3\sqrt{5}}{\sqrt{-6}}, \frac{6}{i^3}, \frac{i^{13}}{i^{12}} \).

• \textbf{Critical Thinking Writing Activity}

You previously discussed the fact that you cannot apply the property \( \sqrt{b^m} = (\sqrt{b})^m \) to this problem \( \sqrt{(-9)^4} \neq (\sqrt{-9})^4 \) and cannot apply the radical product rule \( \sqrt{ab} = \sqrt{a} \sqrt{b} \) to this problem \( (\sqrt{-4})(\sqrt{-9}) \neq \sqrt{-4 \times -9} \). Using what you now know about imaginary numbers, justify that these two statements are truly inequalities and explain why.

**Activity 7: Properties and Operations on Complex Numbers (GLEs 1, 2, 5)**

Students will develop the Complex Number System and develop all operations on complex numbers including absolute value of a complex number.

**Bellringer:** Identify which sets of numbers (Real, Rational, Irrational, Integer, Natural, Whole, Imaginary) contain the following numbers:

1. 4
2. –5
3. 0
4. 5.7
5. 4.1\( \bar{6} \) 3.33333…
6. \( \pi \)
7. \( \frac{1}{2} \)
8. \( \sqrt{2} \)
9. \( 5 + \sqrt{3} \)
10. \( i \)
11. \( \sqrt{-1} \)
12. \( \sqrt{-4} \)
13. \( \sqrt{-8} \)
14. \( 2 + 3i \)

**Activity:**

• Use the Bellringer to define complex numbers as any number in the form of \( a + bi \) in which \( a \) and \( b \) are real numbers and \( i \) is \( \sqrt{-1} \). Redefine the set of Real Numbers as numbers in the form \( a + bi \) where \( b = 0 \), and Imaginary numbers as numbers in the form \( a + bi \) where \( b \neq 0 \). Therefore, if the complex number is \( a + bi \), then the real part is \( a \) and the imaginary \( bi \). The complex conjugate is defined as \( a – bi \).

• \textbf{Complex Property Race}

When creating any new number system, certain mathematical terms must be defined. To review the meaning of these terms in the Real number system and allow students
to define them in the Complex number system, divide students into teams and then divide teams into pairs. Give each pair of students on the team a piece of construction paper or overhead transparency and a different property. Have them define what they think the property is in words and using a + bi and give a real number example and a complex number example without using the book. Have the pair present the property to the class and let the class decides which team wins for each property. The team with the most Best Properties wins a bonus point (or candy, etc.) List of properties and operations: equality, addition, additive identity, additive inverse, subtraction, multiplication, multiplicative identity, squaring, dividing, absolute value, reciprocal, commutative under addition/multiplication, associative under addition/multiplication, closed under addition/multiplication, factoring the difference in 2 perfect squares, factoring the sum of two perfect squares

- Assign a worksheet of problems in which they have to add, subtract, multiply, and divide complex numbers.

- **Critical Thinking Writing Activity**
  Give each pair of students the following activity:

  **Do You Really Know the Difference?**

  State whether the following numbers are real (R) or imaginary (I) and discuss why.

  1) \( i \)
  2) \( i^2 \)
  3) \( \sqrt{-4} \)
  4) \( \sqrt{-2}\sqrt{-5} \)
  5) \( i^n \) if \( n \) is even
  6) the sum of an imaginary number and its conjugate
  7) the difference of an imaginary number and its conjugate
  8) the product of an imaginary number and its conjugate
  9) the conjugate of an imaginary number
  10) the conjugate of a real number
  11) the reciprocal of an imaginary number
  12) the additive inverse of an imaginary number
  13) the multiplicative identity of an imaginary number
  14) the additive identity of an imaginary number

  **Solution:** (1) I  (2) R  (3) I  (4) R  (5) R  (6) R  (7) I  (8) R  (9) I  (10) R  (11) I  (12) I  (13) R  (14) R
Activity 8: Finding Complex Roots of an Equation (GLEs 1, 2, 4, 5, 6, 7, 9, 16)

In this activity, students will find the complex roots of an equation and reinforce the difference in root and zeros using technology.

**Bellringer:** Solve the following equations analytically and write all answers in $a + bi$ form:

1. $x^2 - 16 = 0$
2. $x^2 + 16 = 0$
3. $x^2 + 50 = 0$
4. $(2x - 3)^2 = 18$
5. $(3x - 2)^2 = -24$
6. $x^3 - 28x = 0$
7. $x^3 + 32x = 0$

**Solution:**

1. $± 4 + 0i$
2. $0 ± 4i$
3. $0 ± 5i\sqrt{2}$
4. $\frac{3 ± 3\sqrt{2}}{2} + 0i$
5. $-\frac{2}{3} ± \frac{2i\sqrt{6}}{3}$
6. $±2\sqrt{7} + 0i$
7. $0 ± 4i\sqrt{2}$

**Activity:**

- Have students classify each of the answers of the Bellringer as real or imaginary.
- Have students graph each of the equations in the Bellringer in their graphing calculators and draw conclusions about the number of roots and zeroes of a polynomial. Review the definition of root as the solution to a single variable equation and zero as the $x$ value where $y$ equals zero and reiterate that the $x$ and $y$–axis on the graph are real numbers.
- Review solving polynomials by factoring using the zero property. Have the students predict the number of roots of $x^4 - 16 = 0$, solve it by factoring into $(x + 2)(x - 2)(x^2 + 4) = 0$ and applying the zero property, the predict the number of zeroes and end behavior of the graph of $y = x^4 - 16$.

**Sample Assessments**

**General Assessments**

- The teacher will use Bellringers as ongoing informal assessments.
- The teacher will collect the Little Black Book of Algebra II Properties and grade for completeness at the end of the unit.
- The teacher will monitor student progress using small quizzes to check for understanding during the unit on such topics as the following:
  1. simplifying radicals
  2. adding, subtracting, and multiplying radicals
(3) dividing radicals and rationalizing the denominator
(4) simplifying complex numbers

- The student will demonstrate proficiency on two comprehensive assessments:
  (1) Radicals
  (2) Complex Number System

Activity-Specific Assessments

- **Activity 2:** The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the following rubric:
  
  *Grading Rubric for Critical Thinking Writing Activities*
  
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols
  3 pts./graph - correct graphs (if applicable)
  3 pts./solution - correct equations, showing work, correct answer
  3 pts./discussion - correct conclusion

- **Activity 3:** The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the rubric provided in the assessment for Activity 2.

- **Activity 4:** The teacher will evaluate the Translating Radical Functions – Discovery Worksheet (see activity) using the following rubric:
  
  *Grading Rubric for Discovery Worksheets*
  
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols
  2 pts./graph - correct graphs and equations (if applicable)
  5 pts/discussion - correct conclusions

- **Activities 5, 6 and 7:** The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the rubric provided in the assessment for Activity 2.
Algebra II
Unit 5: Quadratic and Higher Order Polynomial Functions

Time Frame: Approximately six weeks

Unit Description

This unit covers solving quadratic equations and inequalities by graphing, factoring, using the Quadratic Formula and modeling quadratic equations in real-world situations. Graphs of quadratic functions are explored with and without technology using symbolic equations as well as using data plots.

Student Understandings

Students will understand the progression of their learning in Algebra II. They studied first-degree polynomials (lines) in Unit 1 and factored to find rational roots of higher order polynomials in Units 2 and irrational and imaginary roots in Unit 4. Now they can solve real-world application problems that are best modeled with quadratic equations and higher order polynomials going from equation to graph and graph to equation. They will understand the relevance of the zeroes, domain, range, and maximum/minimum values of the graph as it relates to the real-world situation they are analyzing. Students will distinguish between root of an equation and zero of a function and why it is important to find the zeroes of an equation using the most appropriate method. They will also understand how imaginary and irrational roots affect the graphs of polynomial functions.

Guiding Questions

1. Can students graph a quadratic equation and find the zeroes, vertex, global characteristics, domain, and range with technology?
2. Can students graph a quadratic function in standard form without technology?
3. Can students complete the square to solve a quadratic equation?
4. Can students solve a quadratic equation by factoring and using the Quadratic Formula?
5. Can students determine the number and nature of roots using the discriminant?
6. Can students explain the difference in a root of an equation and zero of the function?
7. Can students look at the graph of a quadratic equation and determine the nature and type of roots?
8. Can students determine if a table of data is best modeled by a linear, quadratic, or higher order polynomial function and find the equation?
9. Can students draw scatter plots using real world data and create the quadratic regression equations using calculators?
10. Can students solve quadratic inequalities using a sign chart and a graph?
11. Can students use synthetic division to evaluate a polynomial for a given value and show that a given binomial is a factor of a given polynomial?
12. Can students determine the possible rational roots of a polynomial and use these and synthetic division to find the irrational roots?
13. Can students graph a higher order polynomial with real zeroes?

Unit 5 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
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<tr>
<td><strong>Number and Number Relations</strong></td>
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<tr>
<td>1.</td>
<td>Read, write, and perform basic operations on complex numbers (N-1-H) (N-5-H)</td>
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<tr>
<td>2.</td>
<td>Evaluate and perform basic operations on expressions containing rational exponents (N-2-H)</td>
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<tr>
<td><strong>Algebra</strong></td>
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<tr>
<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>5.</td>
<td>Factor simple quadratic expressions including general trinomials, perfect squares, difference of two squares, and polynomials with common factors (A-2-H)</td>
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<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
</tr>
<tr>
<td>7.</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)</td>
</tr>
<tr>
<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td>9.</td>
<td>Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-4-H)</td>
</tr>
<tr>
<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
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<tr>
<td><strong>Geometry</strong></td>
<td></td>
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<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
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<tr>
<td>19.</td>
<td>Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)</td>
</tr>
<tr>
<td>22.</td>
<td>Explain the limitations of predictions based on organized sample sets of data (D-7-H)</td>
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<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
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<tr>
<td>24.</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
</tbody>
</table>
| 27. | Compare and contrast the properties of families of polynomial, rational,
<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
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<tbody>
<tr>
<td></td>
<td>exponential, and logarithmic functions, with and without technology (P-3-H)</td>
</tr>
<tr>
<td>28.</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
<tr>
<td>29.</td>
<td>Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)</td>
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**Sample Activities**

**Ongoing:** Little Black Book of Algebra II Properties - Quadratic and Higher Order Polynomial Functions

The following is a list of properties to be included in the Little Black Book of Algebra II Properties. Add other items as appropriate.

5.1 **Quadratic Function** – give examples in standard form and demonstrate how to find the vertex and axis of symmetry.

5.2 **Translations and Shifts of Quadratic Functions** - discuss the effects of the symbol $\pm$ before the leading coefficient, the effect of the magnitude of the leading coefficient, the vertical shift of equation $y = x^2 \pm c$, the horizontal shift of $y = (x - c)^2$.

5.3 **Three ways to Solve a Quadratic Equation** – write one quadratic equation and show how to solve it by factoring, completing the square, and using the quadratic formula.

5.4 **Discriminant** – give the definition and indicate how it is used to determine the nature of the roots and the information that it provides about the graph of a quadratic equation.

5.5 **Factors, $x$-intercept, $y$-intercept, roots, zeroes** – write definitions and explain the difference between a root and a zero.

5.6 **Comparing Linear functions to Quadratic Functions** – give examples to compare and contrast $y = mx + b$, $y = x(mx + b)$, and $y = x^2 + mx + b$, explain how to determine if data generates a linear or quadratic graph.

5.7 **How Varying the Coefficients in $y = ax^2 + bx + c$ Affects the Graph** - discuss and give examples.

5.8 **Quadratic Form** – Define, explain, and give several examples.

5.9 **Solving Quadratic Inequalities** – show an example using a graph and a sign chart.

5.10 **Polynomial Function** – define polynomial function, degree of a polynomial, leading coefficient, and descending order.

5.11 **Synthetic Division** – identify the steps for using synthetic division to divide a polynomial by a binomial.

5.12 **Remainder Theorem, Factor Theorem** – state each theorem and give an explanation and example of each, explain how and why each is used, state their relationships to synthetic division and depressed equations.

5.13 **Fundamental Theorem of Algebra, Number of Roots Theorem** – give an example of each theorem.

5.14 **Intermediate Value Theorem** - state theorem and explain with a picture.
5.15 **Rational Root Theorem** – state the theorem and give an example.
5.16 **General Observations of Graphing a Polynomial** – explain the effects of even/odd degrees on graphs, explain the effect of the use of ± leading coefficient on even and odd degree polynomials, identify the number of zeros, explain and show an example of double root.
5.17 **Steps for Solving a Polynomial of 4th degree** – work all parts of a problem to find all roots and graph.

**Activity 1: Why Are Zeros of a Quadratic Function Important?** (GLEs: 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 19, 24, 25, 27, 28, 29)

In this activity the students will plot data that creates a quadratic function and determine the relevance of the zeroes and the maximum and minimum of values of the graph. They will also examine the sign and magnitude of the leading coefficient in order to make an educated guess a regression equation for some data. By looking at real world data first, the symbolic manipulations necessary to solve quadratic equations have significance.

**Bellringer:** One side of a rectangle is four inches less than the other side. Find an expression for the area $A(x)$ of the rectangle.

*Solution:* $A(x) = x(x - 4) = x^2 - 4x$

**Activity:**

- Use the Bellringer to relate second-degree polynomials to the name quadratic equations (*area of a quadrilateral*). Discuss the fact that this is a function and have students find the area if $x = 6$ using the function notation, $A(6)$. (12 in$^2$) Identify this shape as a parabola. Define zeroes at the $x$-value for which $y$ is zero, thus indicating an $x$-intercept. Have the students graph the function on their calculators, find the zeros, and determine the real-world meaning of the zeroes for this problem. (*The length of the side for which the area is zero.*) Discuss both the analytical and calculator way to locate zeroes.

- Use the Bellringer graph to discuss, locate, and define the axis of symmetry. Identify the vertex both analytically and with the use of the maximum function on the calculator. Discuss the local and global characteristics of the function. Have students find the domain and range of the graph on the calculator (*Domain: all real numbers, range: $y \geq -4$*) and find the domain that has meaning for the area problem. ($x > 4$)

- Have students graph the following in their calculators and analyze:
  - Graph $y = x^2$ and $y = -x^2$ and make conjectures about the sign of the leading coefficient.
  - Graph $y = 2x^2$, $y = 4x^2$ and $y = .5x^2$ and make conjectures about the magnitude of the leading coefficient.
  - Graph $y = (x - 3)(x + 4)$ and $y = (x - 1)(x + 6)$ and make conjectures about the zeroes.
Graph \( y = 2(x - 5)(x + 4) \) and \( y = -2(x - 5)(x + 4) \) and make conjectures about the zeroes and leading coefficient.

- **Critical Thinking Writing Activity**
  
  Put the students in groups and give them the following scenario:
  
  A tunnel in the shape of a parabola over a two-lane highway has the following features. It is 30 feet wide at the base and 23 feet high in the center. An 8-foot wide 12-foot high truck wants to go through the tunnel. Determine if it will fit and if so if it can stay in its lane.
  
  (1) Make a sketch of the tunnel on a coordinate plane on your paper with the ground as the \( x \)-axis and the left side of the base of the tunnel at \((2, 0)\). Find two more ordered pairs and graph as a scatter plot in your calculator.
  
  (2) Enter the quadratic equation \( y = a(x - b)(x - c) \) in your calculator substituting your \( x \)-intercepts from your sketch into \( b \) and \( c \). Experiment with various numbers for “\( a \)” to find the parabola that best fits this data. Write your equation on the sketch of your graph.
  
  (3) Find the best method to determine whether the truck will fit and the allowable location of the truck. Explain your answer.
  
  \[ \text{Solution: (1) (32, 0) and (17, 23), (2) } y = -0.1(x - 2)(x - 32), \text{ (3) The truck must travel 4.75 feet from the base of the tunnel. It is 8 feet wide and the center of the tunnel is 15 feet from the base so the truck can stay in its lane.} \]

- Have the students put their equations on the board from the Critical Thinking Writing Activity or enter them into the overhead calculator. Discuss the differences, the relevancy of the zeroes and vertex, and the various methods used to solve the problem. Discuss how to set up the equation from the truck problem to solve analytically. Have the students expand, isolate zero, and find integral coefficients to lead to a quadratic equation in the form \( 0 = ax^2 + bx + c \). Graph this equation and find the zeroes on the calculator. This leads to the discussion of why solve quadratic equations.
  
  \[ \text{Solution: } 12 = -0.1(x - 2)(x - 32), \text{ } x^2 + 34x + 184 = 0 \]

**Activity 2: The Vertex and Axis of Symmetry (GLEs: 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 19, 24, 27, 28, 29)**

In this activity the student will graph a variety of parabolas, discovering the changes that shift the graph vertically, horizontally, and obliquely, and determine the value of the vertex and axis of symmetry.
Bellringer:
(1) Graph \( y = x^2, \ y = x^2 + 4, \) and \( y = x^2 - 9 \) on your calculator, find the zeroes and vertices and write a rule for this type of shift.
(2) Graph \( y = (x - 4)^2, \ y = (x + 2)^2 \) on your calculator, find the zeroes and vertices and write a rule for this type of shift.
(3) Graph \( y = x^2 - 6x \) on your calculator. Find the zeroes on the calculator and use \( \text{CALC 4: maximum} \) to find the vertex. Find the axis of symmetry (zeroes: 0, 6, vertex \((3, -9), \text{axis of symmetry } x = 3\)).
(4) What is the relationship between the vertex and the zeroes? What is the relationship between the vertex and the coefficients of the equation?

Activity:
- Use the Bellringer to begin the development of the formula for finding the vertex of a quadratic function in the form \( f(x) = ax^2 + bx \) by setting \( ax^2 + bx \) equal to 0 to find the zeroes, 0 and \( \frac{-b}{a} \). Find the midpoint between the zeroes at \( \frac{-b}{2a} \) to find the axis of symmetry. Plug in the abscissa to find the ordinate of the vertex \( f\left(\frac{-b}{2a}\right) \). Apply the formula to graphs of functions in the form \( y = ax^2 + bx + c \) and have students practice graphing.
- Critical Thinking Writing Activity: The revenue, \( R \), generated by selling games with a particular price is given by \( R = -15p^2 + 300p + 1200 \). Graph the revenue function and find the price that will yield the maximum revenue. What is the maximum revenue? Explain why the graph is parabolic.

Solution: price = $10, revenue = $2700

Activity 3: Completing the Square (GLEs: 1, 2, 4, 5, 9, 24, 29)

In this activity, students will review solving quadratic equations by factoring and learn to solve quadratic equations by completing the square.

Bellringer:
Solve the following for \( x \):
(1) \( x^2 - 8x + 7 = 0 \),
(2) \( x^2 - 9 = 0 \),
(3) \( x^2 = 16 \),
(4) \( x^2 = -16 \)
(5) \((x - 4)^2 = 25 \)
(6) \((x - 2)^2 = -4 \)

Solutions:
(1) \( x = 7, 1 \), (2) \( x = 3, -3 \), (3) \( x = 4, -4 \), (4) \( x = 4i, -4i \), (5) \( x = 9, -1 \), (6) \( x = 2i + 2, -2i + 2 \)
Activity:
- Use the Bellringer to review the rules for factoring and the zero property for problems 1 and 2. Review the rules for taking the square root of both sides in problems 3 and 4 with real and complex answers.

- Reiterate the difference between the answer for \( \sqrt{16} \) and the solution to the equation \( x^2 = 16 \).

- Discuss the multiple ways of solving problem 5 – expanding, isolating zero, and factoring or taking the square root of both sides, then isolating the variable. Discuss whether these ways can be used to solve problem 6.

- Have students factor the expressions \( x^2 + 6x + 9 \) and \( x^2 –10x + 25 \) to determine what properties of the middle term make these the square of a binomial. Have students find \( c \) so the expressions \( x^2 + 8x + c \) and \( x^2 – 18 x + c \) will be squares of binomials or perfect square trinomials. Name this process “completing the square” and use it to practice solving quadratic equations whose solutions are both real and complex.

- Develop the rule for completing the square when the equation has a leading coefficient.

- Critical Thinking Writing Activity
  Put students in pairs to use completing the square to solve the following application problem:
  
  (1) A farmer has 120 feet of fencing to fence in a dog yard next to the barn. He will use part of the barn wall as one side and wants the yard to have an area of 1000 square feet. What dimensions will the three sides of the yard be?
  (2) Suppose the farmer wants to enclose four sides with 120 feet of fencing. What are the dimensions to have an area of 1000 square feet?
  (3) Approximately how much fencing would be needed to enclose 1000 ft\(^2\) on four sides? Discuss how you determined the answer.

  Solutions:
  (1) \( (x – 30)^2 = 400 \), the three sides of the yard will be 50, 20, and 20 ft.
  (2) There is not enough fencing to enclose 1000 ft\(^2\).
  (3) You would need approximately 126.5 ft of fencing.

Activity 4: The Quadratic Formula (GLEs: 1, 2, 4, 5, 9, 10, 24, 29)

Students will develop the quadratic formula and use it to solve quadratic equations.

Bellringer: Solve the following quadratic equations using any method:

(1) \( x^2 – 25 = 0 \)
(2) \( x^2 + 7 = 0 \)
(3) \( x^2 + 4x =12 \)
(4) \( x^2 + 4x = 11 \)

Solutions: (1) \( x = 5, -5 \), (2) \( x = \pm i\sqrt{7} \), (3) \( x = -6, 2 \), (4) \( x = -2 \pm \sqrt{15} \)
Activity:

• Use the Bellringer to check for understanding of solving quadratic equations by all methods. Emphasize that Bellringer problem 4 must be solved by completing the square because it does not factor.

• Use the process of completing the square to develop the quadratic formula.

• Use the Quadratic Formula to solve all four Bellringer problems. Practice with several more problems.

• Relating Quadratic Formula answers to graphing calculator zeroes
  Have the students put \( y = x^2 + 4x - 7 \) in their calculators, find the zeros, and then use the Quadratic Formula to find the zeros. Then, use the calculator to find the decimal representation for these answers and compare the results.

• Critical Thinking Writing Activity
  John increased the area of his garden by 120 ft\(^2\). The original garden was 12 ft. by 16 ft., and he increased the length and the width by the same amount. Find the exact dimensions of the new garden and approximate the dimensions in feet and inches. Discuss which method you used to solve the problem and why you chose this method.

  \[ x = -14 + 2\sqrt{79}, \text{ dimensions } = -2 + 2\sqrt{79} \times 2 + 2\sqrt{79} = 15\text{ ft. } 9\text{ in. } \times 19\text{ ft. } 9\text{ in} \]

Activity 5: Using the Discriminant and the Graph to Determine the Nature of the Roots (GLEs: 1, 2, 4, 5, 6, 7, 9, 10, 19, 22, 24, 27, 28, 29)

In this activity, students will examine the graphs of shifted quadratic functions, determine the types of roots and zeroes from the graph and from the discriminant, and describe the difference in a root and zero of a function.

Bellringer: Find the zeroes or \( x \)-intercepts of the following functions analytically.

1. \( f(x) = x^2 + 4x - 5 \)
2. \( f(x) = x^2 - 5 \)
3. \( f(x) = (x - 2)^2 \)
4. \( f(x) = x^2 - 3x + 7 \)
5. Graph the above functions on your calculator and describe the differences in the graphs.

  \[ x = -5, 1, x = \pm\sqrt{5}, x = 2, x = \frac{-3 \pm \sqrt{19}}{2} \]

  \( \#1 \) has two real rational roots, \( \#2 \) has two real irrational roots, \( \#3 \) has one real rational root, \( \#4 \) has two imaginary roots
Activity:

• Use the Bellringer to check for understanding for finding zeros and relating them to the graph.

• Have students set up the Quadratic Formula for each of the equations in the Bellringer. Have them determine from the set up what part of the formula decides if the roots are real or imaginary, rational or irrational, one, two or no roots.

• Define \( \sqrt{b^2 - 4ac} \) as the discriminant and have the students develop the rules concerning the nature of the solutions of the quadratic equation. Emphasize the difference in the word root, which can be real or imaginary, and the word zero, which refers to an x-intercept of a graph.

• Define double root and explain what it looks like on a graph.

• Practice predicting solutions using the discriminant.

• Critical Thinking Writing Activity
  Put students in pairs to determine if the following application problem has a solution using a discriminant: The area of a square is 40 more than the area of a rectangle. The length of the rectangle is twice the length of the side of the square and the width of the rectangle is 5 less than the side of the square.
  (1) Use a discriminant to determine if this scenario is possible. If so, find the dimensions of the square. Explain why your solution is possible or not.
  (2) Find a scenario that would make the solution possible, discuss, and solve.

  Solution: (1) not possible, (2) answers vary

Activity 6: Linear Functions versus Quadratic Functions (GLEs: 1, 2, 4, 6, 7, 8, 9, 10, 16, 19, 22, 27, 28)

In this activity the students will discover the similarities and differences in linear and quadratic functions and data.

Bellringer:
Graph without a calculator: \( y = 4x - 8 \) and \( y = x(4x - 8) \). Find the x- and y-intercepts of both and the vertex of the parabola.

Activity:

• Using the Bellringer for discussion, have the students check other pairs of equations in the form \( y = mx + b \) and \( y = x(mx + b) \) to make conjectures.

• Have students graph the Bellringer equations on their calculators and adjust the window to \( x: [1,3] \) and \( y: [-1,1] \). Have them discuss that both graphs look like a line.
with the same \( x \)-intercept and have them determine what type of function is being generated by the data.

- Give the students the following data and ask them which one is a line and why while reviewing the method of finite differences. (See Algebra I, Unit 7, Activity 3.)

<table>
<thead>
<tr>
<th>( x )</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_1 )</td>
<td>-2</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>( y_2 )</td>
<td>-4</td>
<td>0</td>
<td>8</td>
<td>20</td>
<td>36</td>
<td>56</td>
<td>80</td>
<td>108</td>
</tr>
</tbody>
</table>

Solution: \( y_1 \) is a line because the slope, \( \frac{\Delta y}{\Delta x} \), is the same.

- Show students how to find the \( \frac{\Delta y}{\Delta x} \) twice on \( y_2 \) data to prove it is a quadratic function.

Make a scatter plot of the data and find the equation of the best fit line in the form \( y = mx + b \) and the equation of the best fit parabola in the form \( y = x(mx+b) \). Zoom out to determine if the data fits the equations.

- Work several more examples and give some data that is cubic in nature and have the students extend their thinking.

**Activity 7: How Varying the Coefficients in \( y = ax^2 + bx + c \) Affects the Graphs**

(GLEs: 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 19, 27, 28)

In this activity, students will discover how changes in the equation for the quadratic function can affect the graph in order to create a best-fit parabola.

**Bellringer:**

Graph \( y = -4x + 6 \) and \( y = x(-4x + 6) \) without a calculator

**Activity:**
- Use the Bellringer to check for understanding of the relationship between \( y = mx + b \) and \( y = x(mx+b) \) before going on to other changes.

- Divide the students into groups and give them the following worksheet to explore how changes in other coefficients affect the graph.

- **The Changing Parabola – Discovery Worksheet**
  
  (1) Graph \( y = x^2 + 5x + 4 \) which is in the form \( y = ax^2 + bx + c \) (without a calculator). What are the vertex and the \( x \)- and \( y \)-intercepts?

  (2) Predict what will happen to the graphs of form \( y = x^2 + 5x + c \) for the following values of \( c \): \{8, 4, 0, -4, -8\}. Graph on your calculator and check your prediction. What special case occurs at \( c = 0 \)? Explain why the patterns occur.
(3) Predict what will happen to the graphs of form \( y = x^2 + bx + 4 \) for the following values of \( b \): \( \{6, 3, 0, -3, -6\} \). Graph on your calculator and check your prediction. What special case occurs at \( b = 0 \)? Explain why the patterns occur.

(4) Consider \( y = x^2 + bx - 5 = 0 \). Using the results of exercise 3, predict what will happen to the roots of this equation if \( b \to \infty \). Explain the results.

(5) Predict what will happen to the graphs of form \( y = ax^2 + 5x + 4 \) for the following values of \( a \): \( \{2, 1, 0.5, 0, -0.5, -1, -2\} \). Graph on your calculator and check your prediction. What special case occurs at \( a = 0 \)? Explain why the patterns occur.

(6) Consider \( y = ax^2 + x - 5 \). Using the results of exercise 5, predict what happens to the roots of if \( a \to 0 \). Use your calculator to check your prediction. Explain the results.

(7) Consider pairs of equations in the form \( y = mx + b \) and \( y = x^2 + mx + b \). Graph \( y = x^2 + 5x + 4 \) and \( y = 5x + 4 \). How is the line and parabola related? Compare this to pairs of equations in the form \( y = mx + b \) and \( y = x(mx + b) \). Explain why these relationships must hold for all pairs in these forms.

(8) Find all the zeroes by hand for the functions you graphed in exercises 2, 3 and 5 using factoring, completing the square, or the quadratic formula to practice these skills.

**Solutions:**

1. vertex \((-2, -1)\), x-int. = \(-4, -1\), y-int. = 4
2. vertical shifts, If \( c = 0 \), the parabolas passes through the origin.
3. horizontal shifts, If \( b = 0 \), the y-axis is the axis of symmetry.
4. One root is positive and approaches 0 and the other root is negative and approaches \(-\infty \). As \( b \) becomes larger and larger, the constant becomes less significant. If the constant is ignored, the equation becomes \( y = x^2 + bx = 0 \) or \( x(x+b) = 0 \) which has the roots 0 and \(-b\).
5. If \( a = 0 \), the graph is the line \( y = 5x + 4 \).
6. One root is positive and approaches 5 and the other root is negative and approaches \(-\infty \). If \( a = 0 \), the root fails to exist because the equation becomes a line with only one root.
7. The line is tangent to the parabola at the point where the graphs intersect the \( y \)-axis.

**Activity 8: Parabolic Graph Lab (GLEs: 1, 2, 4, 6, 7, 8, 9, 10, 16, 19, 22, 24, 27, 28, 29)**

Students will collect data with a motion detector to determine a quadratic equation for the position of a moving object and use the equation to answer questions.

**Bellringer:**

Graph the following on your calculator and answer the questions:

If the position of a falling object with initial velocity if 50 ft/sec thrown from a height of 100 feet is given by \( f(t) = -16t^2 + 50t + 100 \). Find the maximum height of the object. Find the time the object hits the ground.

**Solution:** maximum height is \( 139 \frac{1}{16} \) ft. and it hits the ground in 4.5106 sec.
Activity:

- Use the Bellringer to check for understanding of the meaning of the vertex and the zeroes.

- Drive the Parabola Lab
  
  Divide the students into groups and have them do this lab:

  Use a CBR or motion detector and CBL, a ramp, and a car (or ball), and push the car up the ramp and let it roll down. Collect data on the motion detector, link your calculator to another calculator to save data, and delete the data that is not in a parabolic shape. Using what you learned about shifts and translations, find the regression equation for the parabolic shape that remains. Answer the following questions in a lab report:

  1. Download the regression equation graph and scatter plot from your calculator into a Word® document on your computer. (If no computer is available have students sketch a graph of the data and regression equation locating the vertex and zeros.)
  2. Before you deleted the outliers, at what time do you think you pushed the car?
  3. What appears to be the vertex of your data? What is the real-world meaning of the vertex of your data? Answer in numbers from your data and units.
  4. What is the regression equation of your parabola?
  5. What is the vertex of the regression equation?
  6. Is your regression equation a good one? Explain your answer using the correlation coefficient and a comparison of the vertex of your data and the vertex of your regression equation.
  7. Define the word interpolation and on your calculator interpolate, using the regression equation to determine where the car was 1.5 seconds after you pushed it. What value did you plug in your equations? Careful – see question #1. Compare this answer to your data. Is it accurate? Explain.
  8. Define the word extrapolation and on your calculator extrapolate to determine where the car was at 6 seconds after you pushed it. What value did you plug in? Is this an accurate assumption? Explain.
  9. Find the zeroes of the regression equation. What is the real-world meaning of these zeroes in relationship to what the motion detector reads?
  10. Describe the mathematical and technical growth you experienced in doing this lab.

Teacher Note: If motion detectors are unavailable, use the following data:

<table>
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<th>time (sec)</th>
<th>0</th>
<th>0.2</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
<th>3.2</th>
<th>3.4</th>
<th>3.6</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.3</td>
<td>0.3</td>
<td>0.3</td>
<td>0.54</td>
<td>0.74</td>
<td>0.9</td>
<td>1.02</td>
<td>1.1</td>
<td>1.14</td>
<td>1.14</td>
<td>1.1</td>
<td>1.02</td>
<td>0.9</td>
<td>0.74</td>
<td>0.54</td>
<td>0.3</td>
<td></td>
</tr>
</tbody>
</table>

Solution: $y = -0.5x^2 + 2.3x - 1.5$
Activity 9: Solving Equations in Quadratic Form (GLEs: 1, 2, 5, 9, 27)

The students will examine equations that are not truly quadratic but in which they can use the same strategies to solve.

Bellringer: Solve the following for \( t \): \( 2t^2 - 4t + 1 = 0 \) (Solution: \( t = \frac{2 + \sqrt{2}}{2} \) and \( t = \frac{2 - \sqrt{2}}{2} \))

Activity:

- Use the Bellringer to review the quadratic formula making sure to have students use the variable \( t \) in the answer and write the two answers separately. Then substitute \((t-3)\) in for \( t \) in the equation and ask them how to solve and how to add 3 to the fractional answers. Show students how to check the answers to prove that they are solutions.

- Define quadratic form and discuss with the students why the following are in quadratic form:
  
  \[
  \begin{align*}
  (1) & \quad h^4 + 7h^2 + 6 = 0 \\
  (2) & \quad x - 3\sqrt{x} - 4 = 0 \\
  (3) & \quad 2(y + 4)^2 + (y + 4) + 6 = 0 \\
  (4) & \quad s^4 + 2s^2 = 0
  \end{align*}
  \]

- Have students work in pairs to solve the problems above making sure to check answers for extraneous roots.

  \[
  Solution: (1) \{\pm i, \pm i\sqrt{6}\}, (2) \{16\}, (3) \left\{\frac{-5}{2}, -6\right\}, (4) \{0, \pm 3i\}
  \]

- Critical Thinking Writing Activity
  In a certain electrical circuit, the resistance of any \( R \) greater than 6 ohms, is found by solving the quadratic equation \((R - 6)^2 = 4(R - 6) + 5\). Show all of your work.
  
  \[
  \begin{align*}
  (1) & \quad \text{Find } R \text{ by solving the equation using quadratic form.} \\
  (2) & \quad \text{Find } R \text{ by first expanding the binomials and factoring.} \\
  (3) & \quad \text{Find } R \text{ by expanding the binomials by using the quadratic formula.} \\
  (4) & \quad \text{Find } R \text{ by graphing } f(R) = (R - 6)^2 - 4(R - 6) - 5 \text{ and finding the zeroes} \\
  (5) & \quad \text{Discuss which of the above methods you like the best and why both solutions for } R \text{ are not used.}
  \end{align*}
  \]

  \[
  Solution: 11 \text{ ohms, 5 ohms is not valid for the initial conditions.}
  \]

Activity 10: Solving Quadratic Inequalities (GLEs: 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 19, 24, 27, 28, 29)

In this activity, students will solve quadratic inequalities using a sign chart as well as by interpreting a graph.
**Bellringer:** Solve the following without a calculator:

1. $8 - 2x > 0$
2. $(x - 4)(x + 3) > 0$

*Solution:* (1) $x > 6$ (2) $x < -3$ or $x > 4$

**Activity:**
- Use the Bellringer to check for students’ understanding that if $ab > 0$, then either both $a$ and $b$ are positive or both $a$ and $b$ are negative. Students will usually forget the second scenario. Graph both solutions on a number line and compare. Discuss the use of *and* or *or*, intersection or union, and how to express the answers in interval notation or set notation.

- Review the zero principle for both equalities and inequalities. Have students develop a method (the sign chart) to find all solutions to quadratic inequalities. Have the students work several examples.

- Have students graph $y = 8 - 2x$ and $y = (x - 4)(x + 3)$ and discuss how these graphs can assist them in solving the inequalities in the Bellringer. What critical numbers on the graphs are important? (*Solution: the zeroes and whether it opens up or down.*) Work several examples using the graph to help solve inequalities.

- **Critical Thinking Writing Activity**
  A truck going through the parabolic tunnel over a two-lane highway has the following features: the tunnel is 30 feet wide at the base and 23 feet high in the center.
  1. Graph your tunnel so that the base is on the $x$-axis and the $x$ intercepts are $\pm 15$.
  2. Find the equation of the parabola and determine the range of heights in order for the vehicle to drive in its own lane if the vehicle is 8-foot wide.
  3. Discuss how you set up the equation for the parabola and how you solved the problem.

  *Solution:* $y = -\frac{23}{225}x^2 + 23$, $0 \text{ ft} < \text{height} < 16\frac{103}{225} \text{ ft}$

**Activity 11: Synthetic Division (GLEs: 1, 2, 5)**

In this activity, students will use synthetic division to divide a polynomial by a first-degree binomial.

**Bellringer:** Divide by hand to simplify the following quotients:

1. $7 \overline{)} 1342$
2. $x - 2 \overline{)} x^3 + 4x^2 - 7x - 14$

*Solution:* (1) $191 \frac{5}{7}$ (2) $x^2 + 6x + 5 - \frac{4}{x - 2}$
Activity:

- Use Bellringer #1 to review elementary grade terminology: 
  \[
  \frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}.
  \]
  Rewrite this rule in Algebra II form: 
  \[
  \frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}
  \]
  and relate to Bellringer problem 2.

- Review the definition of polynomial and the steps for long division, stressing descending powers and missing powers. Have students divide \( \frac{2x^3 + 3x + 100}{x + 4} \).
  
  Solution: \( 2x^2 - 8x + 35 = \frac{40}{x - 4} \)

- Introduce synthetic division illustrating that in the long division problems, the variable is not necessary and if we had divided by the opposite of 4, we could have used addition instead of subtraction. Rework the problems using synthetic division.

- Have students develop the steps for synthetic division:
  1. Set up the coefficients in descending order of exponents.
  2. If a term is missing in the dividend, write a zero in its place.
  3. When dividing by the binomial \( x - c \), use \( c \) as the divisor (\( c \) is the value of \( x \) that makes the factor \( x - c = 0 \)).
  4. When dividing by the binomial \( ax - c \), use \( \frac{c}{a} \) as the divisor. (\( \frac{c}{a} \) is the value of \( x \) that makes the factor \( ax - c = 0 \).)

- Have students practice the use of synthetic division to simplify the following and write the answers in equation form as 
  \[
  \frac{\text{dividend}}{\text{divisor}} = \frac{\text{quotient}}{\text{divisor}} + \frac{\text{remainder}}{\text{divisor}}. 
  \]
  1. \( (2x^3 + 5x^2 - 7x - 12) \div (x + 3) \)
  2. \( (x^4 - 5x^2 - 10x - 12) \div (x + 2) \)

  Solutions:
  1. \( \frac{2x^3 + 5x^2 - 7x - 12}{x + 3} = (2x^2 - x - 4) + \frac{0}{x + 3} \)
  2. \( \frac{x^4 - 5x^2 - 10x - 12}{x + 2} = x^3 - 2x^2 - x + 4 + \frac{4}{x + 2} \)
Activity 12: Remainder and Factor Theorems (GLEs: 1, 2, 5, 6, 8, 9, 25)

In this activity, the students will evaluate a polynomial for a given value of the variable using synthetic division and determine if a given binomial is a factor of a given polynomial.

**Bellringer:** Use the following steps to work problems 1 and 2 below:

(a) Use synthetic division to divide and write the answers in equation form as \[
\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}
\]

(b) Multiply both sides of the equation by the divisor (do not expand) and write in equation form as polynomial = (divisor)(quotient) + remainder) in other words \[P(x) = (x - c)(Q(x)) + \text{Remainder}
\]

(c) What is the remainder?

(d) The divisor is in the form \(x - c\). Find \(P(c)\). What is the relationship between the remainder and \(P(c)\)?

1. \((x^3 + 8x^2 - 5x - 84) \div (x + 5)\)
2. \((x^3 + 8x^2 - 5x - 84) \div (x - 3)\)

**Solutions:**

1. 

(a) \[
\frac{x^3 + 8x^2 - 5x - 84}{x + 5} = x^2 + 3x - 20 + \frac{16}{x + 5}
\]

(b) \(x^3 + 8x^2 - 5x - 84 = (x + 5)(x^2 + 3x - 20) + 16\)

(c) The remainder is 16.

(d) \(P(-5) = 16\). \(P(c)\) equals the remainder.

2. 

(a) \[
\frac{x^3 + 8x^2 - 5x - 84}{x - 3} = x^2 + 11x + 28 + \frac{0}{x + 5}
\]

(b) \(x^3 + 8x^2 - 5x - 84 = (x - 3)(x^2 + 11x + 28) + 0\)

(c) The remainder is 0.

(d) \(P(3) = 0\). \(P(c)\) equals the remainder.

**Activity:**

- **Remainder Theorem**

  Use the Bellringer to have the students develop the Remainder Theorem: If \(P(x)\) is a polynomial and \(c\) is a number, and if \(P(x)\) is divided by \(x - c\), then the remainder equals \(P(c)\).

  o Have students use their calculators to verify the Remainder Theorem.
  o Enter \(P(x) = x^3 + 8x^2 - 5x - 84\) into \(y_1\) and find \(P(-5)\) and \(P(3)\) on the home screen as \(y_1(-5)\) and \(y_1(3)\).
  o Practice: \(f(x) = 4x^3 - 6x^2 + 2x - 5\). Find \(f(3)\) using synthetic division and verify on the calculator. (Solution: \(f(3) = 55\))
  o Have students explain why synthetic division is sometimes called *synthetic substitution.*
• **Factor Theorem**
  o Review the definition of *factor* as two or more numbers or polynomials that are multiplied together to get a third number or polynomial. Have students factor the following quadratic equations:
    
    (1) \( 12 \)
    (2) \( x^2 - 9 \)
    (3) \( x^2 - 5 \)
    (4) \( x^2 + 4 \)
    
    **Solutions:** (1) \( 12 = (3)(4) \), (2) \( x^2 - 9 = (x + 3)(x - 3) \), (3) \( x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5}) \), (4) \( x^2 + 4 = (x + 2i)(x - 2i) \)
  
  o Use Bellringer problem 2 to develop the Factor Theorem: If \( P(x) \) is a polynomial, then \( x - c \) is a factor of \( P(x) \) if and only if \( P(c) = 0 \).
  
  o Relate the Factor Theorem to quadratic equations in earlier activities and have students work the following problem to verify the Factor Theorem:
    
    Factor \( f(x) = x^2 + 3x + 2 \) and find \( f(-2) \) and \( f(-1) \).
    
    **Solution:** \( f(x) = (x + 2)(x + 1) \), \( f(-2) = 0 \) and \( f(-1) = 0 \)
  
  o Using Bellringer problem 2, have the students continue to factor the quotient polynomial, write \( P(x) \) in factored form, and list all the zeroes:
    
    Solution: \( Q(x) = x^2 + 11x + 28 = (x + 7)(x + 4) \), \( P(x) = (x - 3)(x + 7)(x + 4) \), zeroes: \{3, -7, -4\}

• Define \( Q(x) \) as a *Depressed Polynomial* because the powers of \( x \) are one less than the powers of \( P(x) \). Have students understand that the goal is to develop a quadratic depressed equation that can be solved by quadratic function methods. Review the use of the quadratic formula and factoring.

• **Factor Theorem – Discovery Worksheet**
  
  Have students complete the following discovery activity:

  (1) Given one factor of the polynomial, use synthetic division and the depressed polynomial to factor completely.
    
    (1a) \( x + 1; x^3 + x^2 - 16x - 16 \), \( x + 6; x^3 + 7x^2 - 36 \).
    
    (2) Given one factor of the polynomial, use synthetic division to find all the roots of the equation.
    
    (2a) \( x - 1; x^3 - x^2 - 2x + 2 = 0 \), \( x + 2; x^3 - x^2 - 2x + 8 = 0 \)
    
    (3) Given two factors of the polynomial, use synthetic division and the depressed polynomials to factor completely. *Hint:* Use the second factor in the 3rd degree depressed polynomial to get a depressed quadratic polynomial, then factor.
    
    (3a) \( x - 1, x - 3; x^4 - 10x^3 + 35x^2 - 50x + 24 \)
    
    (3b) \( x + 3, x - 4, x^4 - 2x^3 - 13x^2 + 14x + 24 \)

  **Solutions:**
  
  (1a) \( (x + 1)(x - 4)(x + 4) \), \( (x + 6)(x + 3)(x - 2) \),
  
  (2a) \{1, \sqrt{2}, -\sqrt{2}\} \hspace{1cm} (2b) \left\{ -2, \frac{3}{2} + \frac{\sqrt{7}}{2}i, \frac{3}{2} - \frac{\sqrt{7}}{2}i \right\}
  
  (3a) \( (x - 1)(x - 3)(x - 2)(x - 4) \), \( (x + 3)(x - 4)(x - 2)(x + 1) \)
Activity 13: The Rational Root Theorem and Solving Polynomial Equations (GLEs: 1, 2, 5, 6, 7, 8, 9, 25, 27)

In this activity the students will use the rational root theorem and synthetic division to solve polynomial equations.

**Bellringer:** Graph \( f(x) = x^3 + 5x^2 + 12x – 12 \) on your graphing calculator and find all zeroes.

*Solution:* \{-3, -3.646, 1.646\}  *Teacher Note:* Students must ZOOM IN around –3 to find both negative zeroes.

**Activity:**
- Use the Bellringer to review the following concepts from Unit 2:
  1. finding zeroes of a polynomial on a graphing calculator
  2. determining the maximum number of roots for a polynomial equation
  3. remembering what a double zero looks like on a graph
  4. approximating values vs using exact values
- Have the students decide how to use the integer root they found from the graphs and synthetic divisions to find the exact answers of the Bellringer problems.

*Solutions:* \{-3, -1+\sqrt{7}, -1-\sqrt{7}\}

- **Exactly Zero – Discovery Worksheet**
  Put students in pairs to work for this activity.

Graph the following on your calculator and find all exact zeroes:

1. \( f(x) = x^3 + 2x^2 – 10x + 4 \),
2. \( f(x) = x^4 + 2x^3 - 4x^2 – 6x + 3 \),
3. \( f(x) = x^4 – 13x^3 + 81x^2 – 80x + 16 \),
4. \( f(x) = 2x^3 + 7x^2 – x – 2 \),
5. \( f(x) = 3x^3 – 4x^2 – 28x – 16 \)
6. Discuss the process used to find the exact answers.

*Solutions:* (1) \{2, 2 + \sqrt{6}, 2 – \sqrt{6}\}

(2) \{-3, -1 + \sqrt{2}, -1 – \sqrt{2}\}

(3) \{-4, -4, 3 + 2\sqrt{2}, 3 – 2\sqrt{2}\}

(4) \{-\frac{1}{2}, -\frac{3}{2} + \frac{\sqrt{17}}{2}, -\frac{3}{2} – \frac{\sqrt{17}}{2}\}

(5) \{-4, -2, \frac{2}{3}\}

- **Rational Root Theorem**
  - Have students define *rational number*. Possible answers: (1) a repeating or terminating decimal, (2) a fraction \( \frac{p}{q} \) where \( p \) and \( q \) are integers.
  - Have students list the rational roots in each of the above practice problems.
• What is alike about all the polynomials that have integer rational roots?
  Solution: leading coefficient of 1.
• What is alike about all the polynomials that have fraction rational roots?
  Solution: The leading coefficient is the denominator.
  o State the Rational Root Theorem: If a polynomial has integral coefficients, then any
    rational roots will be in the form $\frac{p}{q}$ and $p$ is a factor of the constant and $q$ is a factor
    of the leading coefficient.
  o List all possible rational roots for all the problems above:
    Solutions: (1) $\{\pm 1, \pm 2, \pm 4\}$, (2) $\{\pm 1, \pm 3\}$, (3) $\{\pm 1, \pm 2, \pm 4, \pm 16\}$, (4) $\{\pm 1, \pm 2, \pm \frac{1}{2}\}$,
    (5) $\{\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}\}$
  • Discuss the following theorems and how they apply to the problems above:
    o Fundamental Theorem of Algebra: Every polynomial function with complex
      coefficients has at least one root in the set of complex numbers
    o Number of Roots Theorem: Every polynomial function of degree $n$ has exactly $n$
      complex roots. (Some may have multiplicity.)
    o Complex Conjugate Root Theorem: If a complex number $a + bi$ is a solution of a
      polynomial equation with real coefficients, then the conjugate $a - bi$ is also a solution
      of the equation.
  • Have students decide how to choose which of the many rational roots to use to begin
    synthetic division.
    o Relate back to finding the zeroes on a calculator by entering a lower bound and upper
      bound.
    o Discuss continuity of polynomials.
    o Develop the Intermediate Value Theorem for Polynomials: (as applied to locating
      zeroes). If $f(x)$ defines a polynomial function with real coefficients, and if for real
      numbers $a$ and $b$ the values of $f(a)$ and $f(b)$ are opposite signs, then there exists at
      least one real zero between $a$ and $b$.
  • Critical Thinking Writing Activity
    When you read a mystery, you look for clues to solve the case. Think of solving a
    polynomial equation as a mystery. Discuss all the clues you would look for to find the
    roots of the equation.

Activity 14: Graphing Polynomial Functions (GLEs: 1, 2, 4, 5, 6, 7, 8, 9, 10, 16, 25, 27, 28)

In this activity, the students will tie together all the properties of polynomial graphs learned
in Unit 2 and in the above activities to draw a sketch of a polynomial function with accurate
zeroes and end behavior.
Bellringer: Graph on your graphing calculator. Find exact zeroes and exact roots.

(1) \( f(x) = x^3 - 3x^2 - 5x + 12 \)
(2) \( f(x) = x^4 - 1 \)
(3) \( f(x) = -x^4 + 8x^2 + 9 \)
(4) \( f(x) = -x^3 - 3x^2 \)

Solutions:
(1) zeroes \{-2, 1.5, 3.5\}, roots \{-2, 1.5, 3.5\}
(2) zeroes: \{1, -1\}, roots: \{1, -1, i, -i\}
(3) zeroes: \{3\}, roots: \{3, -3, i, -i\}
(4) zeroes: \{0\}, roots: \{\pm i\sqrt{3}\}

Activity:
- Use the Bellringer to review Unit 2 concepts (end behavior of odd and even degree polynomials, how end-behavior changes for positive or negative leading coefficients) and Unit 5 concepts (the number of roots theorem, rational root theorem, and synthetic division to find exact roots). Discuss what an imaginary root looks like on a graph. (Students in Algebra II will be able to sketch the general graph with the correct zeroes and end-behavior, but the particular shape will be left to Calculus.)

- Ultimate Polynomial – Discovery Worksheet
Put students in pairs to work the following worksheet.

Use the function below to answer all the questions without a calculator:

\[ f(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3 \]

(a) How many roots does the Fundamental Theorem of Algebra guarantee this equation has?
(b) How many roots does the Number of Roots Theorem say this equation has?
(c) List all the possible rational roots.
(d) Use the chart below and the Intermediate Value Theorem to locate the intervals of the zeroes.

<table>
<thead>
<tr>
<th>x</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-.5</th>
<th>0</th>
<th>.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
<th>2.5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>y</td>
<td>25</td>
<td>-12</td>
<td>-18</td>
<td>-11</td>
<td>-3</td>
<td>0</td>
<td>-2</td>
<td>-3</td>
<td>9</td>
<td>52</td>
<td>150</td>
</tr>
</tbody>
</table>

(e) Use the Factor Theorem and depressed equations to find all the roots. Do any roots have multiplicity?
(f) Write the equation totally factored with no fractions and no exponents over 1.
(g) Graph the function without a calculator.
(h) One of your rational roots is a fraction. Discuss the difference in the graph if you use the factor \((x - \frac{a}{b})\) or the factor \((ax - b)\) and which one is correct for this problem.

Solutions:
(a) one complex root
(b) 4
(c) \{\pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 3, \pm \frac{3}{2}, \pm \frac{3}{4}\}
(d) One root at \( x = \frac{1}{2} \) and two other roots in intervals
\(-2 < x < -1.5 \) and \( 1.5 < x < 2 \)
(e) \( \left\{ \frac{1}{2}, \frac{1}{2}, \sqrt{3}, -\sqrt{3} \right\} \) \( \frac{1}{2} \) has a multiplicity of 2
(f) \( f(x) = (2x-1)(2x-1)(x-\sqrt{3})(x+\sqrt{3}) \)
(g) 
(h) \( ax - b \) is correct for this problem. Both equations have the same zeroes, but one has higher and lower minimum points. Since \( f(x) \) has a leading coefficient of 4, my factors must expand to \( 4x^4 + ... \)

Sample Assessments

General Assessments

- The teacher will use Bellringers as ongoing informal assessments.
- The teacher will collect the Little Black Book of Algebra II Properties and grade for completeness at the end of the unit.
- The teacher will monitor student progress using small quizzes to check for understanding during the unit on such topics as the following:
  (1) speed graphing \( y = x^2, y = -x^2, y = x^2 + 4, y = (x + 4)^2 \)
  (2) solving quadratic equations by using the quadratic formula
  (3) solving quadratic equations by completing the square
- The student will demonstrate proficiency on three comprehensive assessments:
  (1) graphing quadratic functions
  (2) solving quadratic equations and inequalities and solving application problems
  (3) use of synthetic division, factor theorem and graphing polynomials
Activity-Specific Assessments

- Activities 1, 2, 3, 4, 5, 9 and 10: The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the following rubric:

  *Grading Rubric for Critical Thinking Writing Activities*
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols
  3 pts./graph - correct graphs (if applicable)
  3 pts./solution - correct equations, showing work, correct answer
  3 pts./discussion - correct conclusion

- Activity 7: The teacher will evaluate The Changing Parabola – Discovery Worksheet (see activity) using the following rubric:

  *Grading Rubric for Discovery Worksheets*
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols
  2 pts./graph - correct graphs and equations (if applicable)
  5 pts/discussion - correct conclusions

- Activity 8: The teacher will evaluate the lab report for “Drive the Parabola” (see activity) using the rubric below:

  *Grading Rubric for Lab Reports*
  10 pts/ question - correct graphs and equations showing all the work
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols

- Activity 12: The teacher will evaluate the Factor Theorem – Discovery Worksheet (see activity) using the rubric shown in the assessment for Activity 7.

- Activity 13: The teacher will evaluate the Exactly Zero – Discovery Worksheet (see activity) using the rubric shown in the assessment for Activity 7.

- Activity 14: The teacher will evaluate The Ultimate Polynomial – Discovery Worksheet (see activity) using the rubric shown in the assessment for Activity 7.
Algebra II
Unit 6: Exponential and Logarithmic Functions

Time Frame: Approximately four weeks

Unit Description

In this unit, students explore exponential and logarithmic functions, their graphs, and applications.

Student Understandings

Students solve exponential and logarithmic equations and graph exponential and logarithmic functions by hand and by using technology. They will understand the speed at which the exponential function increases as opposed to linear or polynomial and determine which type of function best models data. They will understand the meaning of a logarithm of a number and know when to use logarithms to solve exponential functions.

Guiding Questions

1. Can students solve exponential equations with variables in the exponents and having a common base?
2. Can students solve exponential equations not have the same base by using logarithms with and without technology?
3. Can students graph and transform exponential functions?
4. Can students graph and transform logarithmic functions?
5. Can students write exponential functions in logarithmic form and vice versa?
6. Can students use the properties of logarithms to solve equations that contain logarithms?
7. Can students find natural logarithms and anti-natural logarithms?
8. Can students use logarithms to solve problems involving exponential growth and decay?
9. Can students look at a table of data and determine what type of function best models that data and create the regression equation?

Unit 6 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number and Number Relations</td>
<td></td>
</tr>
<tr>
<td>1.</td>
<td>Read, write, and perform basic operations on complex numbers (N-1-H) (N-5-H)</td>
</tr>
<tr>
<td>2.</td>
<td>Evaluate and perform basic operations on expressions containing rational exponents (N-2-H)</td>
</tr>
<tr>
<td>GLE #</td>
<td>GLE Text and Benchmarks</td>
</tr>
<tr>
<td>-------</td>
<td>-------------------------</td>
</tr>
<tr>
<td>3.</td>
<td>Describe the relationship between exponential and logarithmic equations (N-2-H)</td>
</tr>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
</tr>
<tr>
<td>7.</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)</td>
</tr>
<tr>
<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
<tr>
<td>19.</td>
<td>Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)</td>
</tr>
<tr>
<td>22.</td>
<td>Explain the limitations of predictions based on organized sample sets of data (D-7-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>24.</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>27.</td>
<td>Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)</td>
</tr>
<tr>
<td>28.</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
<tr>
<td>29.</td>
<td>Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)</td>
</tr>
</tbody>
</table>

**Sample Activities**

**Ongoing:** **Little Black Book of Algebra II Properties - Exponential and Logarithmic Functions**

The following is a list of properties to be included in the Little Black Book of Algebra II Properties. Add other items as appropriate.

6.1 **Laws of Exponents** – write rules for adding, subtracting, multiplying and dividing values with exponents, raising an exponent to a power, and using negative and fractional exponents.
6.2 **Solving Exponential Equations** – write the rules for solving two types of exponential equations: same base and different bases (e.g., solve $2^x = 8^{x-1}$ without calculator; solve $2^x = 3^{x-1}$ with and without calculator).

6.3 **Exponential Function with Base $a$** – write the definition, give examples of graphs with $a > 1$ and $0 < a < 1$, and locate three ordered pairs, give the domains, ranges, intercepts, and asymptotes for each.

6.4 **Exponential Regression Equation** - give a set of data and explain how to use the method of finite differences to determine if it is best modeled with an exponential equation, and explain how to find the regression equation.

6.5 **Exponential Function Base e** – define $e$, graph $y = e^x$ and then locate 3 ordered pairs, and give the domain, range, asymptote, intercepts.

6.6 **Compound Interest Formula** – define continuous and finite, explain and give an example of each symbol

6.7 **Inverse Functions** – write the definition, explain one-to-one correspondence, give an example to show the test to determine when two functions are inverses, graph the inverse of a function, find the line of symmetry and the domain and range, explain how to find inverse analytically and how to draw an inverse on calculator.

6.8 **Logarithm** – write the definition and explain the symbols used, define common logs, characteristic, and mantissa, and list the properties of logarithms.

6.9 **Laws of Logs** and **Change of Base Formula** – list the laws and the change of base formula and give examples of each.

6.10 **Solving Logarithmic Equations** – explain rules for solving equations, identify the domain for an equation, find log$_2$8 and log$_{25}$125, and solve each of these equations for $x$: log$_2$9 = 2, log$_4$x = 2, log$_4$(x-3)+log$_4$x=1).

6.11 **Logarithmic Function Base $a$** – write the definition, graph $y = \log_ax$ with $a < 1$ and $a > 1$ and locate three ordered pairs, identify the domain, range, intercepts, and asymptotes, and find domain of $y = \log(x^2+7x+ 10)$.

6.12 **Natural Logarithm Function** – write the definition and give the approximate value of $e$, graph $y = \ln x$ and give the domain, range, and asymptote, and locate three ordered pairs, solve $\ln x = 2$ for $x$.

6.13 **Exponential Growth and Decay** - define half-life and solve an example problem, give and solve an example of population growth using $Pe^t$.

**Activity 1: Fractional Exponents (GLEs: 1, 2, 4)**

In this activity, students will review properties of numbers with integral exponents first discussed in Unit 3 and extend them to simplify and evaluate expressions with fractional exponents.

**Bellringer:** Simplify the following and explain the law of exponents used.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>$a^2a^3$</td>
</tr>
<tr>
<td>(2)</td>
<td>$b^7$</td>
</tr>
<tr>
<td>(3)</td>
<td>$(c^3)^4$</td>
</tr>
<tr>
<td>(4)</td>
<td>$2x^5 + 3x^5$</td>
</tr>
</tbody>
</table>

**Solutions:** (1) $a^5$, (2) $b^4$, (3) $c^{12}$, (4) $9x^5$, (5) $8x^3$, (6) $a^2 + 2ab + b^2$, (7) 1, (8) $1/2$
Activity:
- Choose students to write the Laws of Exponents used in the Bellringers on an overhead transparency or on the board. Critique the wording as a class, stressing the need for a common base in problems 1 and 2, a common base and exponent in problem 4, and a common exponent in problem 5.

- Have the students discover the equivalency of the following in their calculators and write a rule for fractional exponents:
  - $\sqrt{5}$ and $5^{\frac{1}{2}}$
  - $\sqrt{6}$ and $6^{\frac{1}{3}}$
  - $\sqrt[4]{2^{3}}$ and $2^{\frac{3}{4}}$

- Have students practice changing radicals to fractional exponents and vice versa using the laws of exponents by simplifying complex radicals. Have students simplify problems such as the following without calculators and use the properties in the Bellringers to simplify similar problems with fractional exponents:
  1. $\left(\frac{1}{100}\right)^{\frac{1}{3}}$
  2. $8^{\frac{1}{3}}$
  3. $625^{\frac{1}{5}}$
  4. $\sqrt[3]{4}$

  Solutions: (1) $\frac{1}{10}$, (2) 2, (3) 5, (4) 8

- Critical Thinking Writing Activity
  1. Simplify $\sqrt{(-9)^2}$.
  2. Simplify $(\sqrt{-9})^2$.
  3. Discuss why the answers to problems 1 and 2 are different.
  4. Discuss why one of the Laws of Exponents, $a^{\frac{b}{c}} = \sqrt[\sqrt{a^b}] = \left(\sqrt[\sqrt{a}]\right)^b$, does not apply to this problem.

  Solutions:
  1. 9
  2. -9
  3. By order of operations, in problem 1 you have to square it first and in problem 2 you have to take the square root first.
  4. This Law of Exponents only works for $a \geq 0$. 

Algebra II ◊ Unit 6 ◊ Exponential and Logarithmic Functions 102
Activity 2: Graphs of Exponential Function (GLEs: 1, 2, 4, 6, 7, 8, 19, 25, 27, 28, 29)

In this activity the students will discover the graph of an exponential function and its domain, range, intercepts, shifts, and effects of differing bases and use the graph to explain irrational exponents.

Bellringer: Graph \( y = x^2 \) and \( y = 2^x \) on your calculator and describe the difference.

Activity:

- Discuss the Bellringer in terms of how fast the functions increase. Show how fast exponential functions increase by the following demonstration:
  
  Place 1 penny on the first square of a checker board, double it and place two pennies on the second square, 4 on the next, 8 on the next, and so forth until the piles are extremely high. Have the students determine how many pennies would be on the last square, tracing to that number on their calculators. Measure smaller piles to determine the height of the last pile and compare it to the distance to the sun, which is 93,000,000 miles.

- Graphing Exponential Functions – Discovery Worksheet

  Have students work in pairs to complete the following discovery worksheet:

  Graph the following functions that are in the form \( y = b^x \) using your calculator, sketch the graphs on your paper, locating the \( y \)-intercept for each function. Then answer the questions 9 through 18.

  (1) \( f(x) = 2^x \)
  (2) \( f(x) = 3^x \)
  (3) \( f(x) = 5^x \)
  (4) \( f(x) = \frac{1}{2}^x \)
  (5) \( f(x) = \frac{1}{5}^x \)
  (6) \( f(x) = \frac{1}{8}^x \)
  (7) \( f(x) = -2^x \)
  (8) \( f(x) = -\frac{1}{2}^x \)

  (9) What point do most of the graphs have in common? Which ones do not have that point in common and what is different about them?
  (10) What happens to the graph as \( b \) increases in problems 1, 2, and 3?
  (11) Describe the graph if \( b = 1 \).
  (12) Describe the difference in graphs for problems 1 through 6 if \( b > 1 \) and if \( 0 < b < 1 \).
  (13) What is the domain of all the graphs?
  (14) What is the range of the graphs?
  (15) Are there any asymptotes? If so what is the equation of the asymptote?
(16) Predict what the graph of \( y = 2^{(x-3)} + 4 \) would look like before you graph on your calculator and explain why.

(17) What is the effect of putting a negative sign in front of \( b^x \)?

(18) Write a short paragraph formulating some properties of graphs of \( f(x) = b^x \).

**Solutions:**

(9) (0,1), problems 7 and 8 do not have this point in common because of the negative sign

(10) graph becomes more steep

(11) horizontal line

(12) \( b > 1 \) increasing, \( 0 < b < 1 \) decreasing

(13) all reals

(14) problems 1 through 6 range: \( y > 0 \); problems 7 and 8 range: \( y < 0 \)

(15) yes, at \( y = 0 \)

(16) The graph is shifted right 3 and up 3

(17) rotates graph on the x-axis

- Examine the graph of \( f(x) = 2^x \) in the Bellringer and discuss its continuity by using the trace function on the calculator to determine \( f\left(\frac{3}{2}\right), f\left(\sqrt{3}\right), \text{and } f\left(2\right) \).

- Discuss irrational exponents with the students and have them apply the Laws of Exponents to simplify the following expressions:

  (1) \( 5^{\sqrt{5}} \cdot 5^{6\sqrt{5}} \)

  (2) \( \frac{6^\sqrt{6}}{6^{\sqrt{2}}} \)

  (3) \( \frac{8^x}{2^x} \)

  (4) \( \frac{2^{\sqrt{5}} \cdot 4^{\sqrt{5}}}{16^{-2^{\sqrt{5}}}} \)

**Solutions:** (1) \( 5^{\sqrt{5}} \), (2) \( 6^{\sqrt{6}} \), (3) \( 4^x \), (4) \( 2^{\sqrt{5}} \)

**Activity 3: Regression Equation for an Exponential Function (GLEs: 2, 4, 6, 7, 8, 10, 19, 22, 27, 28, 29)**

In this activity, the students will enter data into their calculator and change all the parameters for an exponential equation of the form, \( y = Ab^{x-C} + D \), to find the best regression equation and then use the equation to interpolate and extrapolate.
Bellringer: Graph without a calculator. Use what you know about shifts.
(1) \( f(x) = -3^x \)
(2) \( f(x) = 3^{-x} \)
(3) \( f(x) = 3^x - 4 \)
(4) \( f(x) = 3^x - 4 \)
(5) \( f(x) = 5(3^x) \)

Activity:
- Use the Bellringer to check for understanding of translations.
- Exponential Regression Equations – Discovery Worksheet
  Divide students into groups to complete the following discovery worksheet:

  Part I
  (1) Enter the following data into your calculator and determine a regression equation changing the constants \( A \) and \( b \) in \( f(x) = Ab^x \) to find the best equation to fit the data.
  (Do not use the regression feature of the calculator.)

  Wind tunnel experiments are used to test the wind friction or resistance of an automobile at the following speeds.

<table>
<thead>
<tr>
<th>Speed (mph)</th>
<th>Resistance (lbs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>6.4</td>
</tr>
<tr>
<td>21</td>
<td>9.2</td>
</tr>
<tr>
<td>34</td>
<td>17.0</td>
</tr>
<tr>
<td>40</td>
<td>22.4</td>
</tr>
<tr>
<td>45</td>
<td>30.2</td>
</tr>
<tr>
<td>55</td>
<td>59.2</td>
</tr>
</tbody>
</table>

  (2) Write your equation and discuss why you chose the values \( A \) and \( b \).
  (3) When each group is finished, write your equation on the board. Enter all the equations from the other groups into your calculator and vote on which one is the best fit. Discuss why.
  (4) Use the model (regression equation) determined by the class as the best fit to predict the resistance of a car traveling at 50 mph and 75 mph.

  Part II
  (5) Evaluate the following table of data using the method of finite differences to determine which data represents a linear, quadratic, or exponential function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>11</td>
<td>19</td>
<td>18</td>
</tr>
</tbody>
</table>
(6) Discuss what happens in the method of finite differences for an exponential function.

(7) Make a scatter plot and find the regression equations for each. (*Hint: The exponential function is in the form* \( f(x) = b^x + D \)).

(8) Explain the limitations of predictions based on organized sample sets of data.

**Solutions:**

- \( f(x) = 3.493 \cdot (1.05)^x \)
- \( f(50) = 40.058 \)
- \( f(75) = 135.660 \)
- \( f(x) \text{ is linear, } g(x) \text{ is quadratic and } h(x) \text{ is exponential} \)
- \( f(x) \text{ is linear, } g(x) \text{ is quadratic and } h(x) \text{ is exponential} \)
- \( f(x) = 2x + 3, \ g(x) = x^2 + 3, \ h(x) = 2^x + 2 \)

**Activity 4: Exponential Data Research (GLEs: 4, 6, 7, 8, 10, 19, 22, 27, 29)**

This is an out-of-class activity in which the student will find data that is best modeled by an exponential curve.

**Exponential Data Research Project**

This is an individual project and each person must have different data so be the first to print out the data and claim the topic. Possible topics include: US Bureau of Statistics, Census, Stocks, Disease, Bacteria Growth, Investments, Land Value, Animal Population, number of stamps produced each year.

**Directions:**

1. Search the Internet or newspaper and find data that is exponential in nature.
2. Plot the data using at least ten points using either your calculator or an Excel spreadsheet.
3. Print out the table of data and the graph from your calculator or spreadsheet.
4. Find the mathematical model (regression equation) for the data.
5. Compose a relevant question that can be answered using your model and answer it.
6. Write a paragraph (minimum five sentences) about the subject of your study. Discuss any limitations on using the data for predictions. Be sure to include the URL and date of the data.
7. Include all the above information on a poster.
8. Present findings to the class

**Activity 5: Solving Exponential Equations with Common Bases (GLEs: 1, 2, 4, 10)**

In this activity students will use their properties of exponents to solve exponential equations with similar bases.
Bellringer: Graph \( y = 2^{x+1} \) and \( y = 8^{2x+1} \) on your graphing calculator and find the point of intersection.

Solution: \( x = -\frac{2}{5} \)

Activity:
- Define *exponential equation* as an equation in which a variable appears in the exponent and have students discuss a method for solving the Bellringer analytically. They will develop the property, For \( b > 0, b \neq 1, b^{x_1} = b^{x_2} \) if and only if \( x_1 = x_2 \).
- Use the property above to solve the following equations.
  
  \begin{align*}
  \text{(1)} & \quad 3^{x+2} = 6^{2x} \\
  \text{(2)} & \quad 3^{-x} = 81 \\
  \text{(3)} & \quad \left( \frac{3}{2} \right)^{x+1} = \left( \frac{27}{8} \right) \\
  \text{(4)} & \quad 8^x = 4 \\
  \text{(5)} & \quad \left( \frac{1}{27} \right)^x = 81
  \end{align*}

Solutions: \( \text{(1)} \ x = \frac{2}{3}, \ (2) \ x = -4, \ (3) \ x = \frac{1}{2}, \ (4) \ x = \frac{2}{3}, \ (5) \ x = -\frac{4}{3} \)

- Critical Thinking Writing Activity
  1. Solve the two equations \( x^2 = 9 \) and \( 3^x = 9 \)
  2. Discuss the family of equations to which they belong.
  3. Discuss how the equations are alike and how they are different.
  4. Discuss the two different processes used to solve for \( x \).

Activity 6: Inverse Functions and Logarithmic Functions (GLEs: 1, 2, 3, 4, 8, 19, 25, 27)

In this activity, students will review the concept of inverse functions in order to develop the inverse of an exponential function – the logarithmic function.

Bellringer: Find the inverse \( f^{-1}(x) \) of \( f(x) = \frac{2}{x+1} \) and state its domain and range.

Activity:
- Review the concepts of an inverse function – definition, one-to-one correspondence, symmetry about the line \( y = x \), swapping domain and range, swapping abscissa and ordinate in an ordered pair, and finding the inverse analytically.
- Graph \( y = 2^x \) and \( y = x \) on the calculator and use the calculator function, ZOOM, 5:ZSquare. Draw the graph of the inverse on graphing calculator (DRAW, 8: Draw Inv, \( y_1 \)). Discuss the graph of the inverse – domain, range, increasing and decreasing, intercepts, and asymptote.
- Attempt to find the inverse of \( y = 2^x \) analytically and use this discussion to define logarithm and its relationship to exponents.
• Use the concepts of inverse to simplify the following expressions:
  (1) $3^{\log_{3}8}$
  (2) $5^{\log_{5}\sqrt{2}}$
  (3) $\log_{3}3^{17}$
  (4) $\log_{15}15^{\sqrt{3}}$

  Solutions: (1) 8, (2) $\sqrt{2}$, (3) 17, (4) $\sqrt{3}$

• Define common logarithm and evaluate the following logarithmic expressions:
  (1) $\log_{5}125$
  (2) $\log_{10}1000$
  (3) $\frac{1}{4}\log_{16}243$
  (4) $\log_{3}81$
  (5) $\log_{\sqrt{3}}3^{12}$

  Solutions: (1) 3, (2) 3, (3) $-2$, (4) 4, (5) 24

• Critical Thinking Writing Activity
  The value of $\log_{3}16$ is not a number you can evaluate easily in your head. Discuss how you can determine a good approximation.

Activity 7: Graphing Logarithmic Functions (GLEs: 1, 2, 3, 4, 6, 7, 8, 10, 19, 25, 27, 28)

In this activity, students will learn how to graph logarithmic functions, determine the properties of logarithmic functions, and apply shifts and translations.

Bellringer: Evaluate the following:
  (1) $\log_{10}100000$
  (2) $\log_{2}32$
  (3) $\log_{\frac{1}{5}}243$

  Solutions: (1) 5, (2) 5, (3) $\frac{-5}{2}$

Activity:
• Use the Bellringer to check for understanding of evaluating logarithms in different bases.

• Graphing Logarithmic Functions – Discovery Worksheet
  Form pairs of students and give them graph paper to complete the following discovery worksheet:

  (1) Using the equation $f(x) = \log x$, plot points by hand for $x = 1, 10, 100, 0.1, 0.01$ and connect the dots. Discuss the shape of graph and its speed of increasing, domain, range, asymptotes, and intercepts.
  (2) Graph the following functions by hand in the form $f(x) = A \log B (x - C) + D$ using your knowledge of shifts and translations. Label the asymptotes, and $x$- and $y$-intercepts when possible. Have each group put one problem on the board and justify their answer to the class.
(a) $f(x) = 3 \log x$
(b) $f(x) = \log (x - 5)$
(c) $f(x) = \log 3x$
(d) $f(x) = \log x + 2$
(e) $f(x) = -\log x$

(3) Discuss the difficulty in finding the $x$- and $y$-intercepts by hand for logarithms.

(4) Discuss domain restrictions on $g(x) = \log(x - 1)$ and discuss why you cannot find $g(1)$ or $g(0)$.

(5) Graph $y = \log_2 x$ and $\log_3 x$ by locating three ordered pairs and discuss values of $x$ to use when graphing by hand.

(6) Discuss the difference in the graph when changing the base.

Activity 8: Laws of Logarithms and Solving Logarithmic Equations (GLEs: 1, 2, 3, 10)

In this activity, the students will express logarithms in expanded form and as a single log in order to solve logarithmic equations.

Bellringer: Solve for $x$.

(1) $\log_5 25 = x$
(2) $\log_2 x = 3$
(3) $\log_4 16 = 4$
(4) $\log_3 (\log_2 73) = \log_4 x$

Solutions: (1) $x = 2$, (2) $x = 8$, (3) $x = 2$, (4) $\frac{1}{4}$

Activity:

• Use the Bellringer to discuss how to solve different types of logarithmic equations.

• Have the students find the following values in their calculators. When finished, have them compare the answers to draw some conclusions about the Laws of Logs and their relationship to the Laws of Exponents.

(1) $\log 4 + \log 8$
(2) $\log 4 - \log 8$
(3) $\log 32$
(4) $\log 0.5$
(5) $2\log 4$
(6) $\log 16$

• Expand numerous complex logarithmic expressions and combine expanded forms into a single logarithmic expression, then solve equations using these Laws of Logs.

• Critical Thinking Writing Activity

The decibel scale measures the relative intensity of a sound. One formula for the decibel level, $D$, of sound is $D = 10 \log \left( \frac{I}{I_0} \right)$, where $I$ is the intensity level in watts per square meter and $I_0$ is the intensity of barely audible sound.
(1) If the intensity level of a jet is $10^{14}$ watts per square meter times the intensity of barely audible sound ($10^{14}I_0$), what is the decibel level of a jet take-off.

(2) The decibel level of loud music with amplifiers is 120. How many times more intense is this sound than a barely audible sound?

(3) Compare the decibel levels of jets and loud music.

(4) Are there any ordinances in your town about the acceptable decibel level of sound?

Solutions: (1) 140 decibels, (2) $10^{12}I_0$

Activity 9: Solving Exponential Equations with Unlike Bases (GLEs: 1, 2, 3, 10)

Students will use logarithms to solve exponential equations of unlike bases.

Bellringer: Solve for $x$: $3^{2x} = 27^{x+1}$ by hand. (Solution: $x = -3$)

Activity:

- Use the Bellringer to review solving exponential equations which have the same base.

- Have students find $\log_{10}62$ on calculator then change $\log_{10}62 = x$ to the exponential equation $10^x = 62$, noting that this is an exponential equation with different bases (10 and 6). Develop the process for solving exponential equations with different bases using logarithms.

- Use the calculator to find the point of intersection of $y = 3^x$ and $y = 1.5$. Discuss this alternate process for solving the equation $3^x = 1.5$.

- Determine the $\log_2 8$ by hand and $\frac{\log_{10}8}{\log_{10}2}$ on the calculator and formulate a formula for changing the base. Verify the formula by solving the equation $\log_5 7 = x$ by changing it into an exponential equation then solving using logarithms base 10.

- Critical Thinking Writing Activity
  A biologist wants to determine the time $t$ in hours needed for a given culture to grow to 567 bacteria. If the number $N$ of bacteria in the culture is given by the formula $N = 7(2)^t$, find $t$. Discuss the steps used to solve this problem.

  Solution: 6.3 hours
Activity 10: Exponential Growth and Decay (GLEs: 1, 2, 3, 4, 7, 8, 10, 19, 24, 29)

Students will model exponential growth and apply logarithms to solve the problems.

Bellringer:
A millionaire philanthropist walks in class and offers to either pay you one cent on the first day, two cents on the second day, and double your salary every day thereafter for thirty days or to pay you one lump sum of exactly one million dollars. Which choice will you take?

Solution: If you took the first option, after 30 days you would have $10,737,418.23

Activity:
• Instruct students to complete the following exercise.
  Start with 6 Skittles®. Pour out the Skittles®. Assume that the ones with the S showing have had babies and add that many more Skittles® to the cup. Repeat the process until all 50 Skittles® have been used. Create another scatter plot and find the regression equation on your calculator.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (Number you have after adding babies.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

• This activity resembles half-life. Divide the students in groups of four. Give each group a cup of Skittles® (or M & M's®) with approximately 50 candies in each cup. Have students pour out the Skittles® and remove the ones with the S showing as these represent an organism that has contacted a radioactive substance and has died. Ask students to repeat the process until one candy is left and then fill in the following chart. Have them enter this data into their calculators, create a scatter plot, and find a regression equation. When each group has their own data, combine the statistics and find a regression equation for the whole set of data.

<table>
<thead>
<tr>
<th>Year</th>
<th>Population (Number left without S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Activity 11: Compound Interest and Half Life Applications (GLEs: 1, 2, 3, 10, 19, 24)

Students will develop the compound interest and half-life formulas and use them to solve application problems.

Bellringer: If you have $2000 dollars and you earn 6% interest in one year, how much money will you have at the end of a year? Explain the process you used.

Solution: $2120
Activity:
- Use the Bellringer to review the concept of multiplying by 1.06 to get the final amount in a one-step process. Discuss compounding interest semiannually and quarterly. Complete the following chart to develop the compound interest formula, \( A = P\left(1 + \frac{r}{n}\right)^{nt} \), for $2000 invested at 6% APR (annual percentage rate) compounded semiannually (thus 3% each 6 months = 2 times per year).

<table>
<thead>
<tr>
<th>Time years</th>
<th>Do the Math</th>
<th>Developing the Formula</th>
<th>Account Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2000</td>
<td>$2000</td>
<td>$2000.00</td>
</tr>
<tr>
<td>1/2</td>
<td>$2000(1.03)</td>
<td>$2000(1+.06/2)</td>
<td>$2060.00</td>
</tr>
<tr>
<td>1</td>
<td>$2060(1.03)</td>
<td>$2000(1+.06/2)(1+.06/2)</td>
<td>$2121.80</td>
</tr>
<tr>
<td>1.5</td>
<td>$2121.80(1.03)</td>
<td>$2000(1+.06/2)(1+.06/2)(1+.06/2)</td>
<td>$2185.454</td>
</tr>
<tr>
<td>2</td>
<td>$2185.454(1.03)</td>
<td>$2000(1+.06/2)(1+.06/2)(1+.06/2)(1+.06/2)</td>
<td>$2251.01762</td>
</tr>
<tr>
<td>t</td>
<td>$2000(1+.06/2)^{2t}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Assign different rows of students to do the calculations for compounding yearly, quarterly, monthly, and daily. Solve other problems calculating how long it will take to double your money in these situations.

- Define half-life and develop the exponential decay formula, \( A = A_0 \left(\frac{1}{2}\right)^{t/T} \), and use it to solve the following problem:
  A certain substance in the book bag deteriorates from 1000g to 400g in 10 days. Find its half-life.

Activity 12: Natural Logarithms (GLEs: 1, 2, 3, 4, 6, 7, 8, 10, 24, 27)

The students will determine the value of \( e \) and define natural logarithm.

Bellringer: Use your calculator to determine \( \log 10 \) and \( \ln e \). Draw conclusions.

Activity:
- Define \( \ln \) as a natural logarithm base \( e \). Have students do the following activity to discover the approximation of \( e \). Let students use their calculators to complete the following table. Have them put the equation in \( y_1 \) and use the home screen and the notation \( y_1(1000) \) to find the values.

<table>
<thead>
<tr>
<th>( n )</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10,000</th>
<th>100,000</th>
<th>1,000,000</th>
<th>1,000,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left(1 + \frac{1}{n}\right)^n )</td>
<td>205937</td>
<td>207048</td>
<td>2.7169</td>
<td>2.7181</td>
<td>2.7182682</td>
<td>2.718280469</td>
<td>2.718281827</td>
</tr>
</tbody>
</table>
- Define $e$ as the value that this series approaches as $n$ gets larger and larger. It is approximately equal to 2.72 and was named after Leonard Euler in 1750. Stress that $e$ is a transcendental number similar to $\pi$. Although it looks as if it repeats, the calculator has limitations. The number is really 2.71828182845904590… and is irrational.

- Graph $y = \ln x$ and $y = e^x$ and discuss inverses.

- Compare $\left(1 + \frac{1}{n}\right)^n$ to the compound interest formula, $A = Pe^{rt}$, which is derived by increasing the number of times that compounding occurs until interest has been theoretically compounded an infinite number of times. Have students solve problems compounding continuously. Revisit the problem from a previous activity in which the students invested $2000 at 6% APR, but this time compound it continuously for one year. Solution: $2123.67$

- Discuss use of this formula in population growth.

- Critical Thinking Writing Activity
  In 1990 statistical data estimated the world population at 5.3 billion with a growth rate of approximately 1.9% each year.
  (1) Let 1990 be time 0 and determine the equation that best models population growth.
  (2) What will the population be in the year 2010?
  (3) What was the population in 1980?
  (4) In what year will the population be 10 billion?
  (5) Discuss the validity of using the data to predict the future.

  Solution: (1) $A = 5.3e^{0.19t}$, (2) 7.8 billion, (3) 4.4 billion, (4) 2023

Activity 13: Comparing Interest Rates (GLEs: 1, 2, 10, 24)

This is an out-of-class activity. Have students choose a financial institution in town or on the Internet. If possible, have each student in a class chooses a different bank. Have them contact the bank or go online to find out information about the interest rates available for two different types of accounts and how they are compounded. Have students fill in the following information and solve the following problems. When all projects are in, have students report to the class.

Money in the Bank – Data Research Project

Information Sheet: Name of Bank, name of person you spoke to, bank address and phone number or the URL if online, types of accounts, interest rates, and how funds are compounded.
Problem: Create a hypothetical situation in which you invest $500.

(1) Find the graph and regression equation to model two different accounts for your bank.
(2) Determine how much you will have at the end of high school and at the end of college for each account. (Assume you finish high school in one year and college four years later.)
(3) Determine how many years it will take you to double your money for each account.
(4) Determine in which account you will put your money and discuss why.
(5) Display all information on a poster board and report to the class.

Sample Assessments

General Assessments

- The teacher will use Bellringers as ongoing informal assessments.
- The teacher will collect the Little Black Book of Algebra II Properties and grade for completeness at the end of the unit.
- The teacher will monitor student progress using small quizzes to check for understanding during the unit on such topics as the following:
  (1) solving exponential equations with same base
  (2) graphing \( y = e^x \) and \( y = \log x \) with shifts
  (3) evaluating logs such as \( \log_{2}8 \)
- The student will demonstrate proficiency on two comprehensive assessments:
  (1) exponential equations and graphs, evaluating logs, properties of logs and logarithmic graphs
  (2) solving exponential equations with the same base and different bases, and application problems

Activity-Specific Assessments

- Activity 1, 5, 6, 8, 9 and 12: The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the following rubric:

Grading Rubric for Critical Thinking Writing Activity

2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
2 pts. - correct use of mathematical language
2 pts. - correct use of mathematical symbols
3 pts./graph - correct graphs (if applicable)
3 pts./solution - correct equations, showing work, correct answer
3 pts./discussion - correct conclusion
• **Activity 2:** The teacher will evaluate the *Graphing Exponential Functions – Discovery Worksheet* (see activity) using the following rubric:

*Grading Rubric for Discovery Worksheet*

- 2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
- 2 pts. - correct use of mathematical language
- 2 pts. - correct use of mathematical symbols
- 2 pts./graph - correct graphs and equations (if applicable)
- 5 pts./discussion - correct conclusions

• **Activity 3:** The teacher will evaluate the *Exponential Regression Equations – Discovery Worksheet* (see activity) using the rubric for the assessment in Activity 2.

• **Activity 4:** The teacher will evaluate the *Exponential Data Research Project* (see activity) using the following rubric:

*Grading Rubric for Data Research Project*

- 10 pts. - data with proper documentation
- 10 pts. - graph
- 10 pts. - equations, domain, range,
- 10 pts. - real world problem using interpolation and extrapolation, with correct answer
- 10 pts. - poster - neatness, completeness, readability

• **Activity 7:** The teacher will evaluate the *Graphing Logarithmic Functions – Discovery Worksheet* (see activity) using the rubric for the assessment in Activity 2.

• **Activity 13:** The teacher will evaluate the *Money in the Bank - Data Research Project* (see activity) using the rubric for the assessment in Activity 4.
Algebra II
Unit 7: Further Investigations of Functions

Time Frame: Approximately four weeks

Unit Description

This unit ties together all the functions studied throughout the year. It categorizes them, graphs them, translates them, and models data with them.

Student Understandings

The students will understand how the rules affecting change of degree, coefficient, and constants apply to all functions. They will be able to quickly graph the basic functions and make connections between the graphical representation of a function and the mathematical description of change. They will be able to translate easily among the equation of a function, its graph, its verbal representation, and its numerical representation.

Guiding Questions

1. Can students quickly graph lines, power functions, radicals, logarithmic, exponential, step, rational, and absolute value functions?
2. Can students determine the intervals on which a function is continuous, increasing, decreasing, or constant?
3. Can students determine the domains, ranges, zeroes, asymptotes, and global characteristics of these functions?
4. Can students use translations, reflections, and dilations to graph new functions from parent functions?
5. Can students determine domain and range changes for translated and dilated abstract functions?
6. Can students graph piecewise defined functions, which are composed of several types of functions?
7. Can students identify the symmetry of these functions and define even and odd functions?
8. Can students analyze a set of data and match the data set to the best function graph?
### Unit 7 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
</tr>
<tr>
<td>7.</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)</td>
</tr>
<tr>
<td>8.</td>
<td>Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)</td>
</tr>
<tr>
<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Data Analysis, Probability, and Discrete Math</strong></td>
<td></td>
</tr>
<tr>
<td>19.</td>
<td>Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)</td>
</tr>
<tr>
<td>20.</td>
<td>Interpret and explain, with the use of technology, the regression coefficient and the correlation coefficient for a set of data (D-2-H)</td>
</tr>
<tr>
<td>22.</td>
<td>Explain the limitations of predictions based on organized sample sets of data (D-7-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>24.</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>25.</td>
<td>Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>27.</td>
<td>Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)</td>
</tr>
<tr>
<td>28.</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
<tr>
<td>29.</td>
<td>Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)</td>
</tr>
</tbody>
</table>
Sample Activities

Ongoing: Little Black Book of Algebra II Properties - Further Investigations of Functions

The following is a list of properties to be included in the Little Black Book of Algebra II Properties. Add other items as appropriate.

7.1 Basic graphs: Graph and locate \( f(x) \): \( y = x, x^2, x^3, \sqrt{x}, |x|, \frac{1}{x}, \log x, 2^x, \frac{1}{2}x \).

7.2 Continuity – provide an informal definition and give examples of continuous and discontinuous functions.

7.3 Increasing, decreasing, and constant functions – write definitions and draw example graphs such as \( y = \sqrt{9-x^2} \), state the intervals on which the graphs are increasing and decreasing.

7.4 Even and odd functions – write definitions and give examples, illustrate properties of symmetry, and explain how to prove that a function is even or odd (e.g., prove that \( y = x^4 + x^2 + 2 \) is even and \( y = x^3 + x \) is odd).

7.5 General piecewise function – write the definition, and then graph, find the domain and range, and solve \( f(x) = \begin{cases} 2x + 1 & \text{if } x \geq 5 \\ -x^2 & \text{if } x < 5 \end{cases} \) for \( f(4) \) and \( f(7) \).

7.6 Function graph shifts/translations - For problems 1 through 4 below, do the following:

- Explain in words the effect on the graph of a given translation.
- Give an example of the graph of a given abstract function and then the function shifted or transformed (do not use \( y = x \) as your example)
- Explain in words the effect on the domain and range of a given function. Use the domain \([-2,6]\) and the range \([-8,4]\) to find the new domain and range of the translated function.

1) \( f(x + k) \) and \( f(x - k) \), \( f(x) + k \) and \( f(x) - k \)
2) \( f(kx) \), \((|k|<1 \text{ and } |k|>1)\), \( kf(x) \) \((|k|<1 \text{ and } |k|>1)\)
3) \( f(-x) \) and \(-f(x)\)
4) \( f(|x|) \) and \(|f(x)|\)

Activity 1: Basic Graphs and their Characteristics (GLEs: 4, 6, 8, 25, 27)

In this activity the students will work in groups to review the characteristics of all the basic graphs they have studied throughout the year. They will also develop a definition for the continuous, increasing, decreasing, and constant functions.
Bellringer: Graph the following by hand and locate their zeroes and $f(1)$.

1. $f(x) = x$
2. $f(x) = x^2$
3. $f(x) = \sqrt{x}$
4. $f(x) = x^3$
5. $f(x) = |x|$
6. $f(x) = \frac{1}{x}$
7. $f(x) = 2^x$
8. $f(x) = \frac{1}{2} \cdot x$
9. $f(x) = \log x$
10. $f(x) = [x]$ 

Activity:

- Function Calisthenics - Use the Bellringer to review the basic graphs. Then have the students stand up and as you call out a function and ask them to form the shape of the graph with their arms.

- Increasing/decreasing/constant functions
  - Ask students to come up with a definition of **continuity**. (An informal definition of continuity is sufficient for Algebra II.)
  - Then have them develop definitions for **increasing**, **decreasing**, and **constant functions**.
  - Have students look at the abstract graph to the right and determine if it is continuous and the intervals in which it is increasing and decreasing. (Stress the concept that when you ask for intervals, you are asking for intervals of the independent variable, $x$, in this case, and the intervals that are open intervals.)

  **Solution:** Increasing - $(-\infty, -1) \cup (0, \infty)$,

  Decreasing - $(-1, 0)$

  - Have each student graph any kind of graph they desire on the graphing calculator and write down the interval on which the graph is increasing and decreasing. Have students trade calculators with a neighbor and answer the same question for the neighbor’s graph and compare answers.

- Flash that Function
  Divide students into groups of four and give each student ten blank 5 x 7” cards. Have them choose assignments – **Grapher, Symbol Maker, Data Driver, Verbalizer**. Have each member of the group create flash cards of the ten basic graphs in the Bellringer activity, but the front of each will be different based on his/her assignment. (They can use their Little Black Books to review the information.) The front of **Grapher’s card** will have a graph of the function. The front of the **Symbol Maker’s card** will have the symbolic equation of the function. The front of the **Data Driver’s card** will have a table of data that models the function. The front of the **Verbalizer’s card** will have a verbal description of the function. The back of the card will have all the following information: graph, function, the category of parent functions, verbal description, family, table of data, domain, range, asymptotes, intercepts, zeroes, end behavior, and increasing or
decreasing. Once all the cards are complete, have students practice in the group then set up a competition between groups.

Activity 2: Horizontal and Vertical Shifts of Abstract Functions (GLEs: 4, 6, 7, 8, 16, 19, 25, 27, 28)

In this activity, the students will review horizontal and vertical translations, apply them to abstract functions, and determine the effects on the domain and range.

Bellringer: Graph the following without a calculator:

1. \( f(x) = x^2 \)
2. \( f(x) = x^2 + 4 \)
3. \( f(x) = x^2 - 5 \)
4. \( f(x) = (x + 4)^2 \)
5. \( f(x) = (x - 5)^2 \)

Activity:

• Have the students check the Bellringer graphs with their calculators and use the Bellringer to ascertain how much they remember about translations.

• Vertical Shifts: \( f(x) \pm k \)
  - Have the students refer to Bellringer problems 1 through 3 to develop the rule that \( +k \) shifts the functions up and \( -k \) shifts the functions down.
  - Determine if this shift affects the domain or range. (Solution: range)
  - For practice, have students graph the following:
    1. \( f(x) = x^3 \)
    2. \( f(x) = x^3 + 4 \)
    3. \( f(x) = x^3 - 6 \)

• Horizontal Shifts: \( f(x \pm k) \)
  - Have the students refer to Bellringer problems 1, 4, and 5 to develop the rule that \( +k \) inside the parenthesis shifts the function left and \( -k \) shifts the function right, stressing that it is the opposite of what seems logical when shown in the parenthesis.
  - Determine if this shift affects the domain or range. (Solution: domain)
  - For practice, have students graph the following:
    1. \( f(x) = x^3 \)
    2. \( f(x) = (x + 4)^3 \)
    3. \( f(x) = (x - 6)^3 \)
Abstract Vertical and Horizontal Shifts – Discovery Worksheet
Divide students into groups of two or three to work the following discovery worksheet.

Part I
Use the following abstract graph of \( g(x) \) to answer questions 1 through 5.
(1) What is the domain and range of \( g(x) \)?

Draw the graph and find the domain and range of the following:
(2) \( g(x) + 3 \)
(3) \( g(x) - 3 \)
(4) \( g(x + 3) \)
(5) \( g(x - 3) \)

Part II
State the parent function \( f(x) \) and the domain and range of the parent function. Graph the shifted function by hand and state the new domain and range.

\[
(6)\quad j(x) = \sqrt{x+2} - 3
\]
\[
(7)\quad h(x) = |x - 5| + 7
\]
\[
(8)\quad k(x) = \frac{1}{x-3} + 2
\]

Solutions:
(1) \( D: [-5, 6] \quad R: [-3, 8] \)
(2) \( D: \text{same} \quad R: [0, 11] \)
(3) \( D: \text{same} \quad R: [-6, 5] \)
(4) \( D: [-8, 3] \quad R: \text{same} \)
(5) \( D: [-2, 9] \quad R: \text{same} \)
(6) \( f(x) = \sqrt{x} \text{ has } D: x \geq 0, R: y \geq 0, \quad j(x) = \sqrt{x+2} - 3 \text{ has } D: x \geq -2, R: y \geq -3 \)
(7) \( f(x) = |x| \text{ has } D: \text{all reals}, R: y \geq 0, h(x) = |x - 5| + 7 \text{ has } D: \text{all reals}, R: y \geq 7 \)
(8) \( f(x) = \frac{1}{x} \text{ has } D: x \neq 0, R: y \neq 0, \quad k(x) = \frac{1}{x-3} + 2 \text{ has } D: x \neq 3, R: y \neq 2 \)

Finish class with Function Calisthenics - have the students stand up and as you call out a function, have them form the shape of the graph with their arms, but this time call out the basic functions with the vertical and horizontal shifts and negations (e.g., \( x^2 \), \( x^2 + 2 \), \( x^3 \), \( x^3 - 4 \), \( \sqrt{x} \), \( \sqrt{x-4} \), \( \sqrt{x+5} \)).
Activity 3: How Coefficients Change Families of Functions (GLEs: 4, 6, 7, 8, 16, 19, 25, 27, 28)

In this activity the students will determine the effects of a negative coefficient and coefficients with different magnitudes on the graphs and the domains and ranges of functions.

Bellringer: Graph the following on your calculator:

1. \( f(x) = \sqrt{x} \)
2. \( f(x) = -\sqrt{x} \)
3. \( f(x) = \sqrt{-x} \)
4. \( f(x) = \sqrt{9-x^2} \)
5. \( f(x) = 4\sqrt{9-x^2} \)
6. \( f(x) = \frac{1}{2}\sqrt{9-x^2} \)
7. \( f(x) = \sqrt{9-(4x)^2} \)
8. \( f(x) = \sqrt{9-\left(\frac{1}{2}x\right)^2} \)

Activity:

- **Negating the function: \(-f(x)\)**
  - Have the students refer to Bellringer problems 1 and 2 to develop the rule that a negative sign in front of the function rotates the graph about the \(x\)-axis (i.e. all positive \(y\)-values become negative and all negative \(y\)-values become positive).
  - Determine if this affects the domain or range. *(Solution: range)*
  - Practice graphing the following:
    1. \( f(x) = -x^2 \)
    2. \( f(x) = -(\sqrt{x} + 3) \)
    3. If the function \(h(x)\) has a domain \([-4, 6]\) and range \([-3, 10]\), find the domain and range of \(-h(x)\). *(Solution: D: same R: \([-10, 3]\))*

- **Negating the \(x\) within the function: \(f(-x)\)**
  - Have the student refer to Bellringer problems 1 and 3 to develop the rule that the negative sign in front of the \(x\) rotates the graph about the \(y\)-axis (i.e., all positive \(x\) values become negative and all negative \(x\) values become positive).
  - Determine if this affects the domain or range. *(Solution: domain)*
  - Have students practice using the following:
    1. Graph \(f(x) = \log(-x)\)
    2. Graph \(f(x) = \sqrt{-x}\)
    3. If the function \(h(x)\) has a domain \([-4, 6]\) and range \([-3, 10]\), find the domain and range of \(h(-x)\). *(Solution: D: \([-6,4]\) R: same)*

- **Coefficients in front of the function: \(k f(x)\) (\(k > 0\))**
  - Have the students refer to Bellringer problems 4, 5, and 6 to develop the rule:
    - If \(k > 1\), the graph is stretched vertically compared to the graph of \(f(x)\); and if \(0 < k < 1\), the graph is shrunk vertically compared to the graph of \(f(x)\).
  - Ask students to determine if this affects the domain or range. *(Solution: range)*
- Coefficients in front of the $x$: $f(kx)$ ($k > 0$)
  - Have the students refer to Bellringer problems 4, 7, and 8 to develop the rule:
    - If $k > 1$, the graph is shrunk horizontally compared to the graph of $f(x)$; and
    - if $0 < k < 1$, the graph is stretched horizontally compared to the graph of $f(x)$. (Again, when the change is inside the parenthesis, the graph does the opposite of what seems logical.)
  - Ask students to determine if this change affects the domain or range. (Solution: domain)

- Have students practice on the following:
  1. Graph $g(x) = x^2$ and find $g(3)$.
  2. Graph $f(x) = 4(x^2)$ and find $f(3)$.
  3. Compare the answers to $g(3)$ and $f\left(\frac{3}{4}\right)$.
  4. If the function $h(x)$ has a domain $[-4, 6]$ and range $[-3, 10]$, find the domain and range of $h(5x)$.
  5. If the function $h(x)$ has a domain $[-4, 6]$ and range $[-3, 10]$, find the domain and range of $\frac{1}{5}h\left(\frac{1}{5}x\right)$.

  Solution: (1) $g(3) = 9$, (2) $f(3) = 36$, (3) $f(3)$ is 4 time greater than $g(3)$, (4) $D$: same $R$: $[-15, 50]$, (5) $D$: same $R$: $\left[-\frac{3}{5}, 2\right]$
• **Abstract Coefficient Translations – Discovery Worksheet**

Divide students into groups of two or three to work the following discovery worksheet:

**Part I**

Use the following abstract graph of $g(x)$ to answer questions 1 through 5.

1. What is the domain and range of $g(x)$?

   Draw the graph and find the domain and range of the following:

   2. $-g(x)$
   3. $g(-x)$
   4. $2g(x)$
   5. $\frac{1}{2}g(x)$
   6. $g(2x)$
   7. $g(\frac{1}{2}x)$

**Part II**

Some changes do not seem to make a difference. Examine the following situations and answer the questions:

8. Draw the graphs of $f(x) = -x^3$ and $h(x) = (-x)^3$. Find $f(2)$ and $h(2)$.

9. Discuss order of operations. Discuss the difference in the graphs. Explain what effect the parentheses have.

10. Draw the graphs of $f(x) = -x^2$ and $h(x) = (-x)^2$.

11. Discuss the difference in the graphs. Explain what effect the parentheses have.

12. Why do the parentheses affect one set of graphs and not the other?

   **Solution:**
   
   1. $D: [-5, 6] R: [-3, 8]$  
   2. $D$: same $R: [-8, 3]$  
   3. $D: [-6, 5] R$: same,  
   4. $D$: same $R: [-6, 16]$  
   5. $D$: same $R: [-1.5, 4]$  
   6. $D: [-2.5, 3] R$: same,  
   7. $D: [-10, 12], R$: same

• **More Function Calisthenics** – have the students stand up and as you call out a function, have them show the shape of the graph with their arms, but this time have one row make the parent graph and the other rows make graphs with positive and negative coefficients (i.e., $x^2, -x^2, 2x^2, x^3, -x^3, \sqrt{x}, -\sqrt{x}, \sqrt{-x}$).
Activity 4: How Absolute Value Changes Families of Functions (GLEs: 4, 6, 7, 8, 16, 19, 25, 27, 28)

In this activity, students will discover how a graph changes when an absolute value sign is placed around the entire function or just around the variable.

**Bellringer:**
1. Graph \( f(x) = x^2 - 4 \) by hand.
2. Use the graph to solve \( x^2 - 4 \geq 0 \).
3. Use the graph to solve \( x^2 - 4 < 0 \).

Solution: (2) \( x \leq -2 \) or \( x \geq 2 \), (3) \(-2 < x < 2\)

**Activity:**
- Review the definition of absolute value: 
  \[ |x| = \begin{cases} 
  x & \text{if } x \geq 0 \\
  -x & \text{if } x < 0 
  \end{cases} \]
  and review the rules for writing an absolute value as a piecewise function: What is inside the absolute value is both positive and negative. What is inside the absolute value affects the domain.

- **Absolute Value of a Function:** \( |f(x)| \)
  - Have students use the definition of absolute value to write \( |f(x)| \) as a piecewise function:
  \[
  |f(x)| = \begin{cases} 
  f(x) & \text{if } f(x) \geq 0 \\
  -f(x) & \text{if } f(x) < 0 
  \end{cases}.
  \]
  - Have the students write \( |x^2 - 4| \) as a piecewise function and use the Bellringer to simplify the domains.

Solution:
\[
|x^2 - 4| = \begin{cases} 
  x^2 - 4 & \text{if } x^2 - 4 \geq 0 \\
  -(x^2 - 4) & \text{if } x^2 - 4 < 0
  \end{cases} = \begin{cases} 
  x^2 - 4 & \text{if } x \leq -2 \text{ or } x \geq 2 \\
  -(x^2 - 4) & \text{if } -2 < x < 2
  \end{cases}
\]

- Have the students graph the piecewise function by hand reviewing what \(-f(x)\) does to a graph and find the domain and range. (Solution: D: all reals R: \( y \geq 0 \))
- Have the students graph \( f(x) = |x^2 - 4| \) on the graphing calculator.
- Have students develop the rule for graphing the absolute value of a function: Make all \( y \)-values positive. More specifically, keep the portions of the graphs in Quadrants I and II and rotate the graphs in Quadrant III and IV into Quadrants I and II.
- Ask students to determine if this affects the domain or range. (Solution: range)
- Have students practice on the following:
  1. Graph \( g(x) = |x^3| \) and find the domain and range.
  2. Graph \( f(x) = |\log x| \) and find the domain and range.
  3. If the function \( h(x) \) has a domain \([-4, 6]\) and range \([-3, 10]\), find the domain and range of \(|h(x)|\).
  4. If the function \( j(x) \) has a domain \([-4, 6]\) and range \([-13, 10]\), find the domain and range of \(|j(x)|\).
Solutions:
(1) D: all reals, Range: y ≥ 0, (2) D: x > 0, Range: y ≥ 0, 
(3) D: same, R: [0, 10], (4) D: same, R: [0, 13]

• Absolute Value only on the x: f(|x|)
  o Have the students write \( g(x) = (|x| - 4)^2 - 9 \) as a piecewise function.

Solution: \( g(x) = \begin{cases} 
  (x - 4)^2 - 9 & \text{if } x \geq 0 \\
  ((-x) - 4)^2 - 9 & \text{if } x < 0
\end{cases} \).

1. Have the students graph the piecewise function for \( g(x) \) by hand reviewing what the negative only on the x does to a graph.
2. Have students find the domain and range of \( g(x) \). Discuss the fact that negative x-values are allowed and negative y-values may result and that the range is determined by the lowest y-value in Quadrant I and IV, in this case the vertex. (Solution: D: all reals, R: y ≥ 9)
3. Have the students graph \( y_1 = (x - 4)^2 - 9 \) and \( y_2 = (|x| - 4)^2 - 9 \) on the graphing calculator. Turn off \( y_1 \) and discuss what part of the graph disappeared and why.
4. Have students develop the rule for graphing a function with only the x in the absolute value: Graph the function without the absolute value first. Keep the portions of the graph in Quadrants I and IV, discard the portion of the graph in quadrant II and III, and reflect Quadrants I and IV into II and III. Basically, the y-output of a positive x-input is the same y-output of a negative x-input.

  o Have students practice on the following:
    (1) Graph \( y = (|x| + 2)^2 \) and find the domain and range.
    (2) Graph \( y = (|x| - 1)(|x| + 2)(|x| - 3) \) and find the domain and range.
    (3) Graph \( y = \sqrt{|x| - 3} \) and find the domain and range.
    (4) If the function \( h(x) \) has a domain \([-4, 6]\) and range \([-3, 10]\), find the domain and range of \( h(|x|) \)
    (5) If the function \( j(x) \) has a domain \([-8, 6]\) and range \([-3, 10]\), find the domain and range of \( j(|x|) \)

Solutions:
(1) D: all reals, R: y ≥ 4
(2) D: all reals, R: y ≥ 2.061 – cannot be determined without a calculator
(3) D: \( x \leq -3 \) or \( x \geq 3 \), R: y ≥ 0
(4) D: \([-6, 6]\), R: cannot be determined
(5) D: \([-10, 10]\), R: cannot be determined

  o Use the practice problems above to determine if \( f(|x|) \) affects the domain or range.

Solution: \( f(|x|) \) affects both the domain or range. To find the new domain, keep the domain for positive x-values and change the signs to add the negative x-values. The range cannot be determined unless you can determine the maximum and minimum values of y in Quadrants I and IV.
• Abstract Absolute Value Translations – Discovery Worksheet

Divide students into groups of two or three to work the following discovery worksheet:

Part I
Use the following abstract graph of $g(x)$ to answer questions 1 through 5.

1. What is the domain and range of $g(x)$?

Draw the graph and find the domain and range of the following:

2. $|g(x)|$

3. $g(|x|)$

Part II
Some changes do not seem to make a difference. Examine the following situations and answer the questions:

4. Draw the graphs of $t(x) = x^2$, $f(x) = |x^2|$ and $h(x) = |x|^2$. Find $t(-2)$, $f(-2)$ and $h(-2)$.

5. Draw the graphs of $t(x) = x^3$, $f(x) = |x^3|$ and $h(x) = |x|^3$. Find $t(-2)$, $f(-2)$ and $h(-2)$.

6. Which graphs changed and why would one change and not the other?

Solutions:

Activity 5: Functions - Tying It All Together (GLEs: 4, 6, 7, 16, 25, 27, 28)

In this activity, students pull together all the rules of translations, shifts, and dilations.

- **Tying It All Together – Discovery Worksheet**
  Divide students into groups of two or three to work the following discovery worksheet.

**I. Graphing:** Given the graph of the function $f(x)$, match the following shifts and translations.

- (1) $2f(x)$
- (2) $f(2x)$
- (3) $-f(x)$
- (4) $f(-x)$
- (5) $|f(x)|$
- (6) $f(|x|)$
- (7) $f(x) + 4$
- (8) $f(x + 4)$

**II. Domains and Ranges:** Write the new domain and range if $g(x)$ has a domain of [-10, 4] and the range is [-6, 8]. If there is no change, write “no change.”

<table>
<thead>
<tr>
<th></th>
<th>DOMAIN:</th>
<th>RANGE:</th>
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<tbody>
<tr>
<td>1a)</td>
<td>$g(x) + 1$</td>
<td></td>
</tr>
<tr>
<td>b)</td>
<td>$g(x) - 4$</td>
<td></td>
</tr>
<tr>
<td>2a)</td>
<td>$g(x + 1)$</td>
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<td>b)</td>
<td>$g(x - 4)$</td>
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<tr>
<td>3a)</td>
<td>$g(2x)$</td>
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<td>b)</td>
<td>$2g(x)$</td>
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<td>4a)</td>
<td>$g(\frac{1}{2}x)$</td>
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<td>b)</td>
<td>$\frac{1}{2}g(x)$</td>
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<td>5a)</td>
<td>$-g(x)$</td>
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<td>b)</td>
<td>$g(-x)$</td>
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<td>6a)</td>
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<td>b)</td>
<td>$g(</td>
<td>x</td>
</tr>
</tbody>
</table>
Solutions: Functions - Tying It All Together Worksheet


II. (1a) D: no change, R: [–5, 9], (b) D: no change, R: [–10, 4]
(2a) D: [–11, 3], R: no change, (b) D: [–6, 8], R: no change
(3a) D: [–5, 2], R: no change, (b) D: no change, R: [–12, 16]
(4a) D: [–20, 8], R: no change, (b) D: no change, R: [–3, 4]
(5a) D: no change, R: [–8, 6], (b) D: [–4, 10], R: no change
(6a) D: no change, R: [0, 8], (b) D: [–4, 4], R: cannot be determined

Activity 6: More Piecewise Functions  (GLEs: 4, 6, 7, 8, 10, 16, 19, 24, 25, 27, 28, 29)

In this activity, the students will use piecewise functions to review the translations of all basic functions.

- Review the definition of a piecewise function begun in Unit 1: a function made of two or more functions and is written as
  \[ f(x) = \begin{cases} 
  g(x) & \text{if } x \in \text{Domain 1} \\
  h(x) & \text{if } x \in \text{Domain 2}
  \end{cases} 
\]
  where \( \text{Domain 1} \cap \text{Domain 2} = \emptyset \).

- Picture the Pieces – Discovery Worksheet
  Divide students into groups of two or three to work the discovery worksheet.

In problems 1 through 4, graph and state the domain and range, \( x \)- and \( y \)-intercepts, the intervals on which the function is increasing, decreasing or constant and if the function is continuous:

(1) \( f(x) = \begin{cases} 
  \sqrt{x-4} & \text{if } x \geq 4 \\
  x^2 - 10 & \text{if } x < 4
  \end{cases} \)

(2) \( f(x) = \begin{cases} 
  2^{x+1} & \text{if } x \geq -1 \\
  -3|x+4|+10 & \text{if } x < -1
  \end{cases} \)

(3) \( f(x) = \begin{cases} 
  x^3 - 8 & \text{if } x < 0 \\
  \frac{1}{2}x^2 - 1 & \text{if } 0 \leq x \leq 4
  \end{cases} \)

(4) \( f(x) = \begin{cases} 
  |(x-4)^2 - 9| & \text{if } 0 \leq x \leq 8 \\
  \log(x+10) & \text{if } x < 0
  \end{cases} \)

(5) Graph \( h(x) \) and find the \( a \) and \( b \) that makes the function continuous

\( h(x) = \begin{cases} 
  \frac{1}{x-1} & x > 2 \\
  ax + b & 1 \leq x \leq 2 \\
  -\sqrt{x} & x < 1
  \end{cases} \)

(6) Write a piecewise function for the graph of \( g(x) \) below. (Assume all left endpoints are included and all right endpoints are not included.)
(7) Using the graph of $g(x)$ above, draw the graph $h(x) = g(x+5) - 4$ and write its piecewise function.

(8) Using $g(x)$ above, draw the graph of $t(x) = 3g(2x)$ and write its piecewise function.

(9) **Application**
Mary is diabetic and takes long acting insulin shots. Her blood sugar level starts at 100 units at 6:00 a.m. She takes her insulin shot, and the blood sugar increases at a constant rate of 40/hour for three hours. The insulin reaches its peak effect on the blood sugar level and remains constant for five hours. Then it begins to decline for four hours at a constant rate of 30/hr and then remains very low until the next injection the next morning. Let the function $i(t)$ represent the blood sugar level at time $t$ measured in hours from the time of injection. Write a piecewise function to represent Mary’s blood sugar level. Graph $i(t)$ and find the blood sugar level at (a) 7:00 a.m. (b) 10:00 a.m. (c) 5:00 p.m. (d) midnight.

(10) **Application**
Brett is on the ground outside the stadium and throws a baseball to John at the top of the stadium 36 feet above the ground. Brett throws with an initial velocity of 60 feet/sec. It goes above John’s head, and he catches it on the way down. John holds the ball for 5 seconds then drops it to Brett. Graph the function and find a piecewise function that models the height of the ball $s(t)$ over time $t$ in seconds after Brett throws the ball. (Remember the quadratic equation from Unit 5 on position of a free falling object if gravity is $-32 \text{ ft/sec}^2$. $s(t) = -16t^2 + v_0t + s_0$.)
(a) How many seconds after Brett threw the ball did John catch it?
(b) How high did the ball go?
(c) At what time did the ball hit the ground?
Solutions:

(1) D: all reals, R: \( y \geq -10 \), x-intercepts \( \pm \sqrt{10}, 0 \) and \((4,0)\), y-intercepts \((0,-10)\), increasing on \((0,4) \cup (4,\infty)\), decreasing for \( x < 0 \), not continuous,

(2) D: all reals, R: all reals, x-intercept \( -\frac{22}{7}, 0 \), y-intercept \((0,2)\), increasing on \((\infty,-4) \cup (-1,\infty)\), decreasing on \((-4,-1)\), continuous

(3) D: \( x \leq 4 \), R: \((-\infty,-8) \cup \{-1,2,5\}\), no x-intercept, y-intercept \((0,-1)\), increasing \((-\infty,-8)\), constant on \((0,2) \cup (2,4)\), not continuous

(4) D: \((-10,8]\), R: \( y \leq 9 \), x-intercepts \((-9,0),(1,0),(7,0)\), y-intercepts \((0,7)\), increasing on \((-10,0) \cup (1,4) \cup (7,8)\), decreasing \((0,1) \cup (9,7)\), not continuous

(5) \( a = 2 \) and \( b = -7 \)

(6) \( g(x) = \begin{cases} 
  x + 8 & -6 \leq x < 0 \\
  (x - 2)^2 & 0 \leq x < 4 \\
  6.5 & 4 \leq x < 9
\end{cases} \)

(7) \( h(x) = \begin{cases} 
  x + 9 & -11 \leq x < -5 \\
  (x + 3)^2 - 4 & -5 \leq x < -1 \\
  2.5 & -1 \leq x < 4 \\
  6x + 24 & -3 \leq x < 0
\end{cases} \)

(8) \( g(x) = \begin{cases} 
  12(x - 1)^2 & 0 \leq x < 2 \\
  19.5 & 2 \leq x < 4.5
\end{cases} \)

(9i) \( i(t) = \begin{cases} 
  40t + 100 & 0 \leq t \leq 3 \\
  220 & 3 \leq t \leq 8 \\
  -30t + 460 & 8 \leq t \leq 12 \\
  100 & 12 \leq t \leq 24
\end{cases} \) (9a) 140, (b) 220, (c) 130, (d) 100

(10) \( s(t) = \begin{cases} 
  -16t^2 + 60t & 0 \leq t < 3 \\
  36 & 3 \leq t < 8 \\
  -16(t - 8)^2 + 36 & 8 \leq t < 9.5
\end{cases} \) (10a) 3 seconds, (b) 56.25 ft (c) 9.5 seconds

Activity 7: Symmetry of Graphs (GLEs: 4, 6, 7, 8, 16, 25, 27, 28)

In this activity, students will discover how to determine if a function is symmetric to the y-axis, the origin, or other axes of symmetry.
**Bellringer:** Graph without a calculator:

1. \( f(x) = (x)^2, f(-x) = (-x)^2, \) and \(-f(x) = -x^2\)
2. \( f(x) = |x|, f(-x) = |-x|, \) and \(-f(x) = -|x|\)
3. \( f(x) = \log x, f(-x) = \log (-x), \) and \(-f(x) = -\log x\)
4. \( f(x) = 2^x, f(-x) = 2^{-x}, \) and \(-f(x) = -(2^x)\)
5. \( f(x) = x^3, f(-x) = (-x)^3, \) and \(-f(x) = -(x^3)\)

**Activity:**
- Use the Bellringer to discuss the relationships between, \( f(x), f(-x) \) and \(-f(x)\).
- Have students practice with the following:
  6. \( f(x)=\sqrt{x}, f(-x)=\sqrt{-x}, \text{ and } -f(x)=-\sqrt{x} \)
  7. \( f(x)=\frac{1}{x}, f(-x)=\frac{1}{-x}, \text{ and } -f(x)=\frac{-1}{x} \)
  8. \( f(x)=x, f(-x)=-x, \text{ and } -f(x)=-(x) \)

**Even Functions:** Ask the students which of the basic functions in the Bellringer have the property that \( f(-x) = f(x) \) and what kind of symmetry they have in common. (Solution: problems 1 and 2, both symmetric to the y-axis) Define these as **even functions** and note that does not necessarily mean that it has an even power such as in problem 2.

**Odd Functions:** Ask the students which of the basic functions in the Bellringer have the property that \( f(-x) = -f(x) \) and what kind of symmetry they have in common. (Solution: problems 5, 6, 8, and 9, symmetric to the origin) Define these as **odd functions**.

**Neither even nor odd:** Have students graph \( y = x^3 + 1 \) and note that just because it has an odd power does not mean it is an odd function. Ask the students which of the basic functions do not have any symmetry and are said to be neither even nor odd. (Solution: problems 3, 4, and 7)

- Consider the following table of values and determine which of the functions may be even, odd, or neither.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( h(x) )</th>
<th>( s(x) )</th>
<th>( t(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3)</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>(-4)</td>
</tr>
<tr>
<td>(-2)</td>
<td>3</td>
<td>(-4)</td>
<td>5</td>
<td>5</td>
<td>3</td>
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<tr>
<td>(-1)</td>
<td>4</td>
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<td>4</td>
<td>(-4)</td>
<td>(-2)</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>undefined</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>(-5)</td>
<td>2</td>
<td>4</td>
<td>(-2)</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>(-5)</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>(-6)</td>
<td>6</td>
<td>(-6)</td>
<td>(-4)</td>
</tr>
</tbody>
</table>

*Solution: Even: \( f(x) \) and \( t(x) \), Odd: \( g(x) \) and \( s(x) \), Neither: \( h(x) \)*
• Proving even/odd: Discuss whether the seven sets of ordered pairs above are enough to prove that a function is even or odd. Have students see that seven sets of ordered pairs are not sufficient to prove a function as even or odd. For example in \( h(x), \ h(-3) = h(3), \) but the rest did not work. In order to prove whether a function is even or odd, the student must substitute \((-x)\) for every \(x\) and determine if \(f(-x) = f(x)\) or if \(f(-x) = -f(x)\) or neither.

• Have students determine if the following functions are even or odd:
  1. \( f(x) = x^4 + 3x^2 + 5 \)
  2. \( f(x) = 4x^3 + x \)
  3. \( f(x) = |x|^3 + 5 \)
  4. \( f(x) = |x^3| \)
  5. \( f(x) = \sqrt{|x|} \)
  6. \( f(x) = \log |x| \)
  7. \( f(x) = 2^{|x|} \)
  8. \( f(x) = 5x^3 + x^2 \)

  Solution: Even - #1, 3, 4, 5, 6, Odd - #2, Neither - #8

• Critical Thinking Writing Activity
  Discuss other symmetry you have learned in previous units such as the axis of symmetry in a parabola or an absolute value function and the symmetry of inverse functions. Give some examples and find the lines of symmetry.

Activity 8: History, Data Analysis, and Future Prediction of City Statistics (GLEs: 4, 6, 8, 10, 19, 20, 22, 24, 28, 29)

This activity culminates the study of the ten families of functions. Have the student decide which function will best represent certain data. Have them collect data for the past ten years concerning statistics for their city, trace the history of the statistics, evaluate the economic impact, create a PowerPoint® presentation of the data including pictures, history, economic impact, spreadsheet graph of regression line and equation, and future predictions.

• Review with the students how to find a regression equation on their calculators.

• Modeling to Predict the Future - Data Research Project
  Have students divide into pairs and assign the following project.

  Research: Have students work on this in groups of two or elect to work alone. Ask each group to choose a different topic concerning statistical data for their city. Have them collect the data, analyze the data, research the history of the data, and take relevant pictures with the digital camera.

  Calculator/Computer Data Analysis: Have students load the data into their graphing calculators link their graphing calculators to the computer, download the data into a spreadsheet, and create a graph and regression equation of the data points. Then have them use the regression coefficient to determine if the function they chose is reliable.
Interpolation and Extrapolation: Using critical thinking skills concerning the facts, have students make predictions for the next five years and explain the limitations of predictions.

Presentation: Have students create a PowerPoint® presentation including the graph, digital pictures, economic analysis, historical synopsis, and future predictions. Link all the PowerPoint® presentations together and load them on a zip disk for a presentation in the class.

Journal: Finally, ask each student to write a journal entry indicating what he/she learned mathematically, historically, and technology and express his/her opinion of how to improve the project.

Approximate Length of the project: The students will utilize one to two weeks of individual time in research and two to three days of class time for analysis and computer use.

Final Product: Each group must submit:
1) A disk containing the PowerPoint® presentation with these slides.
   Slide 1: Introduction of the topic with a picture.
   Slide 2: History and economics of the topic
   Slide 3: Scatter point graph of the data, curve, and regression equation, regression coefficient.
   Slide 4: Prediction for the next year if the trend continues.
   Slide 5-6: Any other pertinent info, your data, URL for links to other sites for additional information, or another data comparison. Include resources used to find data. (*Teacher note: This should not be necessarily on the last slide.*)
2) A print out of the slides in the presentation.
3) A print out of the data with the cited resources.
4) Release forms signed by all people in the photographs.

Sample Assessments

General Assessments

- The teacher will use Bellringers as ongoing informal assessments.
- The teacher will collect the Little Black Book of Algebra II Properties and grade for completeness at the end of the unit.
• The teacher will monitor student progress using small quizzes to check for understanding during the unit on such topics as the following:
   (1) speed in graphing basic graphs
   (2) vertical and horizontal shifts
   (3) coefficient changes to graphs
   (4) absolute value changes to graphs
   (5) even and odd functions

• The student will demonstrate proficiency on one comprehensive assessment about translations and shifts of functions and graphing piecewise functions.

**Activity-Specific Assessments**

- **Activity 1**: The teacher will evaluate the Flash That Function flash cards for accuracy and completeness.

- **Activity 2**: The teacher will evaluate the Abstract Vertical and Horizontal Shifts – Discovery Worksheet (see activity) using the following rubric:
  
  **Grading Rubric for Discovery Worksheet**
  
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols
  2 pts./graph - correct graphs and equations (if applicable)
  5 pts./discussion - correct conclusions

- **Activity 3**: The teacher will evaluate the Abstract Coefficient Translations – Discovery Worksheet (see activity) using the rubric provided in the assessment for Activity 2.

- **Activity 4**: The teacher will evaluate the Abstract Absolute Value Translations – Discovery Worksheet (see activity) using the rubric provided in the assessment for Activity 2.

- **Activity 5**: The teacher will evaluate the Tying It All Together – Discovery Worksheet (see activity) using the rubric provided in the assessment for Activity 2.

- **Activity 6**: The teacher will evaluate the Picture the Pieces – Discovery Worksheet (see activity) using the rubric provided in the assessment for Activity 2.
• **Activity 7**: The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the following rubric:

*Grading Rubric for Critical Thinking Writing Activity*

- 2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
- 2 pts. - correct use of mathematical language
- 2 pts. - correct use of mathematical symbols
- 3 pts./graph - correct graphs (if applicable)
- 3 pts./solution - correct equations, showing work, correct answer
- 3 pts./discussion - correct conclusion

• **Activity 8**: The teacher will evaluate the Modeling to Predict the Future - Data Research Project (see activity) using the following rubric:

*Grading Rubric for Data Research Project*

- 10 pts. - data with proper documentation
- 10 pts. - graph
- 10 pts. - equations, domain, range,
- 10 pts. - real-world problem using interpolation and extrapolation, with correct answer
- 10 pts. - *PowerPoint*® presentation - neatness, completeness, readability, release forms (if applicable)
- 10 pts. - journal
Time Frame: Approximately four weeks

Unit Description

This unit focuses on the analysis and synthesis of graphs and equations of conic sections and their real-world applications.

Student Understandings

The study of conics helps students relate the cross curriculum concepts of art and architecture to math. They will define parabolas, circles, ellipses, and hyperbolas in terms of the distance of points from the foci and describe the relationship of the plane and the double-napped cone that forms each conic. Students identify various conic sections in real-life examples and in symbolic equations. Students solve systems of conic and linear equations with and without technology.

Guiding Questions

1. Can students use the distance formula to define and generate the equation of each conic?
2. Can students complete the square in a quadratic equation?
3. Can students transform the standard form of the equations of parabolas, circles, ellipses, and hyperbolas to graphing form?
4. Can students identify the major parts of each of the conics from their graphing equations and graph the conic?
5. Can students formulate the equations of each of these conics from their graphs?
6. Can students find real-life examples of these conics, determine their equations, and use the equations to solve real-life problems?
7. Can students identify these conics given their stand and graphing equations?
8. Can the students predict how the graphs will be transformed when certain parameters are changed?
## Unit 8 Grade-Level Expectations (GLEs)

<table>
<thead>
<tr>
<th>GLE #</th>
<th>GLE Text and Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algebra</strong></td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)</td>
</tr>
<tr>
<td>5.</td>
<td>Factor simple quadratic expressions including general trinomials, perfect squares, difference of two squares, and polynomials with common factors (A-2-H)</td>
</tr>
<tr>
<td>6.</td>
<td>Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)</td>
</tr>
<tr>
<td>7.</td>
<td>Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)</td>
</tr>
<tr>
<td>9.</td>
<td>Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-4-H)</td>
</tr>
<tr>
<td>10.</td>
<td>Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)</td>
</tr>
<tr>
<td><strong>Geometry</strong></td>
<td></td>
</tr>
<tr>
<td>15.</td>
<td>Identify conic sections, including the degenerate conics, and describe the relationship of the plane and double-napped cone that forms each conic (G-1-H)</td>
</tr>
<tr>
<td>16.</td>
<td>Represent translations, reflections, rotations, and dilations of plane figures using sketches, coordinates, vectors, and matrices (G-3-H)</td>
</tr>
<tr>
<td><strong>Patterns, Relations, and Functions</strong></td>
<td></td>
</tr>
<tr>
<td>24.</td>
<td>Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)</td>
</tr>
<tr>
<td>27.</td>
<td>Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)</td>
</tr>
<tr>
<td>28.</td>
<td>Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)</td>
</tr>
<tr>
<td>29.</td>
<td>Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)</td>
</tr>
</tbody>
</table>

### Sample Activities

**Ongoing:** Little Black Book of Algebra II Properties - Conic Sections

The following is a list of properties to be included in the Little Black Book of Algebra II Properties. Add other items as appropriate.

8.1 **Circle** – write the definition, provide examples of both the standard and graphing forms of equations and show how to graph them, and provide a real-life example in which circles are used.

8.2 **Parabola** – write the definition, give the two forms for the equation of a parabola and show how to graph them, find the vertex from the equation and from the graph, give examples of the equations of both vertical and horizontal parabolas and their
8.3 **Ellipse** – write the definition, write two forms for the equation of an ellipse and graph each, locate and identify foci, vertices, major and minor axes, explain the relationship of \(a, b,\) and \(c,\) and provide a real-life example in which an ellipse is used.

8.4 **Hyperbola** – write the definition, write the two forms for the equation and draw their graphs, identify vertices, identify transverse and conjugate axes and provide an example of each, explain the relationships between \(a, b,\) and \(c,\) find foci and asymptotes, and give a real-life example in which a hyperbola is used.

8.5 **Conic Sections** – define each, explain the derivation of the names, and draw each as a slice from a cone.

8.6 **Degenerate cases of conics** – give examples of equations for each and draw the picture representations from cones.

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**Activity 1: Deriving the Equation of a Circle (GLEs: 4, 5, 7, 9, 10, 15, 16, 27, 28)**

In this activity, students will review the concepts of Pythagorean theorem and the distance formula studied in Algebra I in order to derive the equation of a circle from its definition.

**Bellringer:**

1. Draw a right triangle with sides 6 and 7 and find the length of the hypotenuse.
2. Find the distance between the points \((x, y)\) and \((1, 3)\).
3. Define a circle.

**Solutions:**

\[
\begin{align*}
(1) & \quad \sqrt{85}, \\
(2) & \quad \sqrt{(x-1)^2 + (y-3)^2}, \\
(3) & \quad \textit{Set of all points in a plane equidistant from a fixed point}
\end{align*}
\]

**Activity:**

- Have students use the information in the Bellringer to derive the graphing form of the equation of a circle with the center on and off origin. Discuss the relationship of Unit 7’s translated functions to this formula. \((x - h)^2 + (y - k)^2 = r^2.\)

- Have students expand the graphing form of a circle with center, not at the origin, to derive the standard form of an equation of a circle. \(Ax^2 + By^2 + Cx + Dy + E = 0\) where \(A = B.\)

- Have students examine the relationships between the square of a binomial and the expanded form. Have students use the method of completing the square to transform the standard form of a circle to graphing form in order to graph the circle.

- Discuss degenerate cases of a circle:

  1. If equation is in graphing form and \(r^2 = 0,\) then the graph is a point, the center.
2. If equation is in graphing form and \( r^2 \) is negative, then the graph is the empty set.

- Have students determine how to graph a circle on their graphing calculators. This will include a discussion of functions, radicals, and ZOOM Square to set the window so the graph looks circular.

- Have students bring in pictures of something in their real-life world with a circular shape.

**Activity 2: Circles - Algebraically and Geometrically (GLEs: 9, 10, 16, 24, 28)**

In this activity, students will review geometric properties of a circle and equations of lines to find equations of circles and apply to real-life situations.

**Bellringer:**

1. Draw a circle and draw a tangent, secant, and chord for the circle and define each.
2. What is the relationship of a tangent and radius and the radius perpendicular to a chord?
3. Find the equation of a line perpendicular to \( y = 2x \) and through the point (6, 10).

**Solutions:**

- **(2) The tangent is perpendicular to the radius at the point of tangency.**
- **A radius perpendicular to a chord also bisects the chord.**
- **(3) \( y = -\frac{1}{2}x + 13 \)**

**Activity:**

- **Circles – Discovery Worksheet**

Put students in groups to compare answers to the Bellringer and work the following discovery worksheet.

Solve the following problems which deal with finding the equation of a circle:

1. Find the equation of the circle with center \((4, 5)\) passing through the point \((-2, 3)\). Graph the points and the circle.
2. Find the equation of the circle with the endpoints of the diameter at \((2, 6)\) and \((-4, -10)\). Graph the points and the circle.
3. Find the equation of the circle with center \((4, -3)\) and tangent to the \(x\)-axes.
4. Find the equation of the circle with center \((4, -3)\) and tangent to the \(y\)-axes.
5. Find the equation of the circle with center \((4, -3)\) and tangent to the line \( y = -\frac{3}{4} \).
6. Find the equation of the circle passing through the points \((5, 3)\) and \((5, 9)\) and has a radius = 5.
(7) Use the pictures brought in Activity 1 to find the equation of one of the circles. Write and solve a story problem in which an equation would be necessary.

Solutions:
(1) \((x - 5)^2 + (y - 4)^2 = 40\)
(2) \((x + 1)^2 + (y + 2)^2 = 73\)
(3) \((x - 4)^2 + (y + 3)^2 = 36\)
(4) \((x - 4)^2 + (y + 3)^2 = 16\)
(5) \((x - 4)^2 + (y + 3)^2 = 0, degenerate case because center is on the line\)
(6) \((x - 1)^2 + (y + 6)^2 = 25\)

Activity 3: Developing Equations of Parabolas (GLEs: 4, 5, 6, 7, 9, 10, 15, 16, 24, 27, 28)

In this activity, students will apply the concept of distance to the definition of a parabola to derive the equations of parabolas, to graph parabolas, and to apply them to real-life situations.

Bellringer: Graph the following by hand:
(1) \(y = x^2\)
(2) \(y = x^2 + 6\)
(3) \(y = (x + 6)^2\)
(4) \(y = x^2 + 2x - 24\)

Activity:
- Use the Bellringer to review the graphs of parabolas as studied in Unit 5 on quadratic functions. Review finding the vertex using \(\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)\) and finding the zeroes.
- Define a parabola in terms of distance from focus and directrix, identifying these terms on a sketch. Demonstrate this definition using the website, www.explorelearning.com.
- Discuss real-life parabolas such as if a ray of light or a sound wave travels in a path parallel to the axis of symmetry and strikes a parabolic dish, it will be reflected to the focus where the receiver is located in satellite dishes, radio telescopes, and reflecting telescopes. More examples of parabolas can be seen at the website, http://www.maa.org/mathland/mathtrek4_13_98.html.
- Parabolas – Discovery Worksheet
  Divide students in pairs. Give each pair two pieces of graph paper and a piece of string to complete the discovery worksheet:
  1) On the first piece of graph paper, locate the focus at \((8, 4)\) and the directrix at \(y = 2\).
  2) Use the string to plot enough points that satisfy the definition of a parabola to create a general parabolic shape.
  3) Discuss the definition of parabola and how to use the definition to find the equation of the parabola.
4) Label one of the points (x, y) and apply the definition and distance formulas to determine the equation of the parabola in graphing form isolating the y.

5) Find the vertex of the parabola. Use the leading coefficient \(a\) from the equation in problem 4 and the vertex and write the equation in graphing form
\[y = a(x - h)^2 + k.\]

6) Enter both of the equations in your calculator to determine if they are coincident.

7) On the next piece of graph paper, locate the focus at (8, 4) and the directrix at \(x = 2\).

8) Use your string to plot enough points to satisfy the definition of a parabola and create a general parabolic shape.

9) Label one of the points (x, y) and apply the definition and distance formula to determine the equation of the parabola by isolating the \(x\).

10) Find the vertex of the parabola. Use the leading coefficient \(a\) from the equation in problem 9 and the vertex to write the equation in graphing form \(x = a(y - k)^2 + h\).

11) Enter both of the equations in your calculator to determine if they are coincident.

12) Determine the distance from the vertex to the focus in Part I. What is the distance from the vertex to the focus in Part II? How is the distance from the vertex to the focus related to the leading coefficient?

13) Find the vertex and the focus in the following equations. Graph each after locating two more pairs of points:

   (a) \(y = \frac{1}{16}(x - 2)^2 - 3\)  
   (b) \(x = 2(y + 3)^2 + 5\)

   (c) What effect do changes in the distance to the focus make in the graph?

14) Graph the following on your graphing calculator and tell how \(a\) determines the shape of the graph:

   (a) \(y = 2x^2 + 4x + 5\)
   (b) \(y = -2x^2 + 4x + 5\)
   (c) \(y = 0.5x^2 + 4x + 5\)

15) Determine how to graph horizontal parabolas on a graphing calculator by isolating the \(y\) and graphing the positive and negative radical. Graph the following:

   (a) \(x = 3y^2 - 2\)  
   (b) \(x = -3y^2 - 2\)  
   (c) \(y = 0.5y^2 + 4y + 5\)  
   (d) \(x = ay^2 + by + c\) What happens when a positive and negative \(a\) are used in the equation \(x = ay^2 + by + c\)?

16) Complete this application problem.

   A satellite is 18 inches wide and 2 inches at its deepest part. What is the equation of the parabola? (Hint: Locate the vertex at the origin and write the equation in the form \(y = ax^2\).) Where should the receiver be located to have the best reception? Hand in a graph and its equation showing all work. Be sure to answer the question in a complete sentence and justify the location. (If you have an old satellite dish, use the dimensions on that to find the location of the receiver.)

**Solutions:**

\(4) \sqrt{(x-8)^2 + (y-4)^2} = y - 2 \Rightarrow y = \frac{1}{4}x^2 - 4x + 19\)

\(5) \text{vertex (8, 3), } y = \frac{1}{4}(x-8)^2 + 3\)
(9) \( \sqrt{(x-8)^2 + (y-4)^2} = x - 2 \Rightarrow x = \frac{1}{12} y^2 - \frac{3}{4} y + \frac{16}{3} \)

(10) \( x = \frac{1}{12} (y - 4)^2 + 4 \)

(12) In Part I, the distance was 1 and in Part II, the distance was 3. The leading coefficient is the reciprocal of 4 times the distance.

(13) (a) Vertex: (2, -3), Focus: (2, 1) (b) Vertex: (–5, 3), Focus: \( \left( \frac{41}{8}, -3 \right) \)

(c) The farther the focus is from the vertex, the narrower the graph.

(14) For vertical parabolas, the use of a positive value for “a” makes the graph open up while using a negative value for “a” makes the graph open down. The smaller the value of “a”, the wider the parabola is.

(15) For horizontal parabolas, the use of positive “a” makes the graph open to the right and the use of negative “a” makes the graph open to the left.

y = \frac{1}{18} x^2, The receiver should be located 4½ inches above the vertex.

Activity 4: Discovering the Graphing Form of the Equation of an Ellipse (GLEs: 4, 5, 7, 9, 10, 15, 16, 27, 28)

In this activity, students will apply the definition of an ellipse to sketch the graph of an ellipse and discover the relationships between the lengths of the focal radii and axes of symmetry. They will also find examples of ellipses in the real world.

Bellringer: Draw an isosceles triangle with base = 8 and legs = 5. Find the length of the altitude. (Solution: 3)

Activity:
- Define ellipse and ask for some examples of ellipses in the real world such as the orbits of the earth around the sun.

- Ellipses – Discovery Worksheet
  Divide students into groups of three. Give each group a piece of graph paper glued to a piece of cardboard with two points on one of the axes evenly spaced from the origin and a piece of string with tacks at each end. Some of the groups will have the same two points with different length string, and some will have the same length string with different points or on a different axis. Use lengths that create Pythagorean triples for this discovery lesson. On the back of each cardboard will be the equation of the ellipse that will be sketched. Have the groups complete the following discovery worksheet:
(1) Use the definition of ellipse to sketch its graph. Use the pins as two of the vertices of the triangle at the given points (called the foci) and a pencil in the third vertex of the triangle. Move the pencil around the foci top.

(2) Since the string represents the sum of the focal radius, find the length of the string using units on your graph paper.

(3) Determine the length of the longest axis of symmetry (the major axis). How does this relate to the sum of the focal radii?

(4) Draw an isosceles triangle with the base on the major axis and the vertices of the base on the foci. The third vertex will lie at the end of the minor axis. Label the lengths of the legs of the triangle and the length of the base. Find the length of the altitude and the length of the shortest axis of symmetry (the minor axis).

(5) Label $\frac{1}{2}$ the major axis as $a$, $\frac{1}{2}$ the minor axis as $b$, and the distance from the center of the ellipse to the focus as $c$. Relate the length of the hypotenuse of the isosceles triangle to the length of $\frac{1}{2}$ the major axis (these are equal) and let the students find a relationship between $a$, $b$, and $c$.

(6) Look on the back of the cardboard to find the equation of your ellipse. How is the information you found related to this equation?

(7) Tape your graph to the board and write the equation below it.

- Use the graphs on the board to draw conclusions about the location of major and minor axes, relationships with focus and focal radii, and the general form for the graph of an ellipse with center at origin.

- Discuss how the graphing form will change if the center is moved away from the origin (relate to translations done previously).

- Demonstrate the definition of ellipse. Have students use the website, www.explorelinking.com, to discover what the distance between foci does to the shape of the ellipse.

- Critical Thinking Writing Activity
  Each group will be assigned one real-life application to research - find pictures and discuss the importance of the foci (e.g., elliptical orbits, machine gears, optics, telescopes, sports tracks, lithotripsy, whisper chambers).

Activity 5: Equations of Ellipses in Standard Form (GLEs: 4, 5, 7, 9, 10, 15, 16, 24, 27, 28)

In this activity, students will determine the standard form of the equation of an ellipse and complete the square to transform the equation of an ellipse from Standard to Graphing Form.

Bellringer: Graph $\frac{(x-2)^2}{25} + \frac{(y+3)^2}{9} = 1$. Find the foci and expand the equation so there are no fractions and zero is isolated.
Activity:
• Use the Bellringer to check for understanding of graphing and finding foci.

• Use the expanded equation in the Bellringer to have students determine the general characteristics of the standard form of the equation of an ellipse. Compare the standard form of an ellipse to the standard forms of equations of circles and parabolas.

• Have students determine how to transform the standard form into the graphing form of an ellipse by completing the square.

• Discuss degenerate cases of an ellipse.

• Have students give their reports on the real-life application assigned in Activity 4.

Activity 6: Determining the Equations and Graphs of Hyperbolas (GLEs: 4, 6, 5, 7, 9, 10, 15, 16, 27, 28)

In this activity, students will apply what they have learned about ellipses to the graphing of hyperbolas.

Bellringer:
Determine which of the following equations is a circle, parabola, line, hyperbola or ellipse.

1. (1) 9x² + 16y² + 18x – 64y – 71 = 0
(2) 9x + 16y – 36 = 0
(3) 9x² + 16y – 36 = 0
(4) 9x – 16y² – 36 = 0
(5) 9x² + 9y² – 36 = 0
(6) 9x² – 4y² – 36 = 0

Solutions: (1) ellipse, (2) line, (3) parabola, (4) parabola, (5) ellipse, (6) hyperbola

Activity:
• Use the Bellringer to check for understanding in problems 1 through 5.

• Students will be unfamiliar with the equation in problem 6. Have the students graph the two halves on their graphing calculators by isolating y and reiterating the concept that the calculator is a function grapher.


• Have students transform the equation in problem 5 into the graphing form of an ellipse and graph it. Then have them transform the equation in problem 6 in the same way and determine the relationships of the numbers in the equation to the graph on the calculators.

• Define transverse and conjugate axes and find their lengths.
• Graph, define, and find the equations of the asymptotes.

• Locate the foci in problem 6. Have students draw a right triangle with a right angle at the center and the other two vertices at one end of the major axis and one end of the minor axis. Relate the length of the hypotenuse to the length of the segment from the center of the hyperbola to the focus (these are equal). Label \( \frac{1}{2} \) the transverse axis as \( a \), \( \frac{1}{2} \) the conjugate axis as \( b \), and the distance from the center of the hyperbola to the focus as \( c \). Let the students determine the relationship between \( a \), \( b \), and \( c \).

• Have the students graph \( 9y^2 - 4x^2 = 36 \) on their calculators and determine how the graph is different from the graph generated by the equation in problem 6.

• Develop the rules for graphing a hyperbola with the center on the origin and with the center not at the origin.

• Transform equations in standard form to equations in graphing form and graph.

• Discuss degenerate forms of the hyperbola.

• Locate the foci in problem 6. Have students draw a right triangle with the right angle at the center and the other two vertices at one end of the major axis and one end of the minor axis. Relate the length of the hypotenuse to the length of the segment from the center of the hyperbola to the focus (these are equal). Label \( \frac{1}{2} \) the transverse axis “\( a \)”, \( \frac{1}{2} \) the conjugate axis “\( b \)”, and the distance from the center of the hyperbola to the focus “\( c \)”. Let the students determine the relationship between \( a \), \( b \), and \( c \).

• Discuss the applications of a hyperbola: the path of a comet often takes the shape of a hyperbola, the use of hyperbolic (hyperbola-shaped) lenses in some telescopes, the use of hyperbolic gears in many machines and in industry, the use of the hyperbolas in navigation since sound waves travel in hyperbolic paths, etc. Some very interesting activities using the hyperbola are available at: http://www.geocities.com/CapeCanaveral/Lab/3550/hyperbol.htm.

Activity 7: Saga of the Roaming Conic (GLEs: 7, 15, 16, 24, 27, 28)

This can be an open or closed-book quiz or in-class or at-home creative writing assignment making students verbalize the characteristics of a particular conic.

Bellringer:
Graph the following pairs of equations on your calculator and determine what the size of the coefficients does to the shape of the graph.
(1) \( y = x^2 \) and \( y = 9x^2 \)
(2) \( 2x^2 + y^2 = 1 \) and \( 9x^2 + y^2 = 1 \)
(3) \( x^2 - y^2 = 1 \) and \( 9x^2 - y^2 = 1 \)
Activity:
- Discuss answers to the Bellringer.

- Critical Thinking Writing Activity
  Saga of the Roaming Ellipse - Give each student one sheet of paper with a full size ellipse, hyperbola or parabola drawn on it and the following directions.
  You are an ellipse (or parabola or hyperbola). Your owner is an Algebra II student who moves you and stretches you. Using all you know about yourself, describe what is happening to you while the Algebra II student is doing his/her homework. You must include ten facts or properties of an ellipse (or parabola or hyperbola) in your discussion.

Activity 8: Comparison of all Conics and the Double-Napped Cone (GLEs: 5, 7, 9, 10, 15, 16, 27, 28)

In this activity, students will compare and contrast all conics – their equations, their shapes, their degenerate forms, their relationships of the plane and double-napped cone that forms each conic, and their applications.

Bellringer:
The following are degenerate cases of conics. Describe the graph.
(1) $2x^2 + y^2 + 6 = 0$
(2) $x^2 + y^2 + 4x - 6y + 13 = 0$
(3) $x^2 - 6x - y^2 + 9 = 0$
(4) $3x^2 + x = 0$
(5) $y^2 = 4$

Solutions: (1) point, (2) point, (3) two intersecting lines, (4) two parallel vertical lines (5) two parallel horizontal lines

Activity:
- Use the Bellringer to check for understanding of recognizing possible conics and their degenerate cases.

- Students often think that a parabola and half of a hyperbola are the same. Give them the equations $\frac{y^2}{8} - \frac{x^2}{16} = 1$ and $y = 0.057941x^2 + 3$, which both have a vertex of (0, 3) and pass through the point $(8, \sqrt{3})$. Have them graph the equations on their calculators zooming in and out to see the differences. Have them find the points of intersection on the calculator.
• **Conics and the Double Napped Cone Lab**

A plane intersecting a double-napped cone can be used to determine each conic and its degenerate case. Place students in groups of six and give each group two cone-shaped pieces of Styrofoam®, a piece of cardboard with graph paper pasted to it, and a plastic knife. Give each group a conic and an application problem for that conic. Each member of the group will have a responsibility:

1. Student A will cut one Styrofoam® cone in the shape of the conic.
2. Student B will sketch the conic formed when cutting the Styrofoam on the plane (cardboard with graph paper).
3. Student C will determine the equation of the graph.
4. Student E will determine how to cut the second cone of Styrofoam® to create the degenerates cases of the conic.
5. Student F will work the application problem.
6. Student G will present the findings to the class.

**Activity 9: Solving Systems of Equations Involving Conics (GLEs: 5, 6, 7, 9, 10, 15, 16, 28)**

Is this activity, students will review the processes for solving systems of equations begun in the unit on Systems of Linear Equations and apply some of these strategies to solving systems involving conics.

**Bellringer:** Graph $y = 3x + 6$ and $2x - 6y = 9$ and find their point of intersection.

**Solution:** $\left(-\frac{45}{16}, \frac{39}{16}\right)$

**Activity:**

- Use the Bellringer to determine if the students remember that finding a point of intersection and solving a system of equations are synonymous. Review solving systems by substitution and elimination (addition).
- Give the students the equations $x^2 + y^2 = 25$ and $y = x - 1$ and ask them to determine the best method to solve. Give them other systems that have no solutions, one solution, and two solutions.
- Give students problems which require simultaneous solving of two conic equations. Have students graph the equations first to determine how many points of intersection exist and have them solve them simultaneously using the most appropriate method.
- Put the students in pairs and give each pair five application problems involving systems of conic equations.
Activity 10: Graphing Art Project (GLEs: 4, 6, 7, 9, 10, 15, 16, 24, 27, 28)

In this Graphing Art project, students will analyze equations to synthesize graphs and then analyze graphs to synthesize equations. The students will draw their own pictures composed of familiar functions, write the equation of each part of the picture finding the points of intersection, and learn to express their creativity mathematically.

Bellringer: Graph the following equations on the same graph paper.

Solution: These equations will produce a picture of a heart.

<table>
<thead>
<tr>
<th>EQUATIONS</th>
<th>RESTRICTIONS ON DOMAINS/RANGES</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y = -\frac{1}{9}(x-6)^2 + 14$</td>
<td>D: [0, 12]</td>
</tr>
<tr>
<td>$y = -\frac{1}{9}x^2 - \frac{4}{3}x + 10$</td>
<td>D: [-12, 0]</td>
</tr>
<tr>
<td>$y = -2x + 34$</td>
<td>D: [12, 13]</td>
</tr>
<tr>
<td>$2x - y = -34$</td>
<td>D: [8, 10]</td>
</tr>
<tr>
<td>$x = 13$ or $x = -13$</td>
<td>R: [5, 10]</td>
</tr>
</tbody>
</table>

Activity:
- This culminating activity follows the unit on conics but involves other functions. It is necessary to review the graphs of lines and absolute value relations, the writing of restricted domains in various forms, and finding points of intersection. The Bellringer models the types of answers that will be expected in the next part of the activity.

- Materials and Supplies: Samples of the sheets needed for this activity can be found in the February, 1995, issue of Mathematics Teacher in an article by Fan Disher entitled “Graphing Art” reprinted in Using Activities from the Mathematics Teacher to Support Principles and Standards, (2004) NCTM.
  
  Sheet 1 – Practice in Graphing
  Sheet 2 – Graphing a Sailboat
  Sheet 3 – Graphing Art Project Directions
  Sheet 4 – Creating a Graph-Equation Blanks
  Sheet 5 – Student Evaluation
  Overhead projector-graph transparencies
  Graph paper with x-axis labeled –14 to 14 and y-axis labeled –18 to 18
  Graphing calculators.
  Optional: Math Type and Graphmatica computer software.

- Divide students into five member cooperative groups and give them Sheet 2 – Graphing a Sailboat, and the enlarged Sheet 4 – Creating a Graph-Equation Blanks. Have group members to determine the equation of each part of the picture and the restrictions on either the domain or range. This group work will promote some very interesting discussions concerning the forms of the equations and how to find the restrictions.

- The students are now ready to begin the individual portion of their projects. Have each student draw a picture, analyze each line and curve, and write the symbolic equations.
represented by the graph. In Sheet 3–Graphing Art Project Directions, the students are instructed to use graph paper either vertically or horizontally to draw a picture containing only graphs of lines, parabolas, and absolute values. The picture must contain at least two lines, all four conics, at least two absolute-value graphs, and two more of any type. One of the absolute value graphs or one of the parabolas must have a horizontal axis of symmetry. On Sheet 4–Creating a Graph - Equation Blanks, the student will record a minimum of ten equations, one for each graph. There is no maximum number of equations, which gives individual students much flexibility; the poorer students can draw the basic picture and equations and achieve while the creative students can draw more complex pictures.

- At this point, this is now an out-of-class project in which the students are monitored halfway through, using a rough draft. Have students turn in final copies of pictures and equations, and their self-evaluations (Sheet 5). After all the equations for have been checked for accuracy, appoint an editor from the class to oversee the compilation of the graphs and equations into a booklet to be distributed to other mathematics teachers to be used in their classes. The students enjoy seeing their names and creations in print and gain a feeling of pride in their creations.

Sample Assessments

General Assessments

- The teacher will use Bellringers as ongoing informal assessments
- The teacher will collect the Little Black Book of Algebra II Properties and grade for completeness at the end of the unit.
- The teacher will monitor student progress using a small quiz after each conic to check for understanding.
- The students will demonstrate proficiency on two comprehensive assessments:
  (1) circles and parabolas
  (2) all conic sections

Activity-Specific Assessments

- Activity 2: The teacher will evaluate the Circles – Discovery Worksheet (see activity) using the following rubric:
  Grading Rubric for Discovery Worksheet
  2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  2 pts. - correct use of mathematical language
  2 pts. - correct use of mathematical symbols
  2 pts./graph - correct graphs and equations (if applicable)
  5 pts./discussion - correct conclusions
• **Activity 3**: The teacher will evaluate the Parabolas – Discovery Worksheet (see activity) using the rubric provided in the assessment for Activity 2.

• **Activity 4**: The teacher will evaluate the Ellipses – Discovery Worksheet (see activity) using the rubric provided in the assessment for Activity 2.

• **Activities 4 and 7**: The teacher will evaluate the Critical Thinking Writing Activity (see activity) using the following rubric:

  **Grading Rubric for Critical Thinking Writing Activity:**
  
  - 2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  - 2 pts. - correct use of mathematical language
  - 2 pts. - correct use of mathematical symbols
  - 3 pts./graph - correct graphs (if applicable)
  - 3 pts./solution - correct equations, showing work, correct answer
  - 3 pts./discussion - correct conclusion

• **Activity 8**: The teacher will evaluate the Conics and the Double Napped Cone Lab (see activity) using the following rubric:

  **Grading Rubric for Lab Report**
  
  - 10 pts/ question - correct graphs and equations showing all the work
  - 2 pts. - answers in paragraph form in complete sentences with proper grammar and punctuation
  - 2 pts. - correct use of mathematical language
  - 2 pts. - correct use of mathematical symbols

• **Activity 10**: The teacher will evaluate the Graphing Art project using several assessments along the way to check progress.

  1. The group members will assess each other’s rough drafts to catch mistakes before the project is graded for accuracy.
  2. The teacher will evaluate the final picture and equations using the following rubric:

     - 75% - graphs and equations
     - 10% - correct types of equations
     - 10% - domain restrictions
     - 5% - appearance

     bonus points - complexity and uniqueness.

• **(3)** The teacher will evaluate the opinion journal to decide whether to change or modify the unit for next year.